

Reconstructing phase dynamics of oscillator networks

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We generalize our recent approach to the reconstruction of phase dynamics of coupled oscillators from data [B. Kralemann *et al.*, Phys. Rev. E **77**, 066205 (2008)] to cover the case of small networks of coupled periodic units. Starting from a multivariate time series, we first reconstruct genuine phases and then obtain the coupling functions in terms of these phases. Partial norms of these coupling functions quantify directed coupling between oscillators. We illustrate the method by different network motifs for three coupled oscillators and for random networks of five and nine units. We also discuss nonlinear effects in coupling. © 2011 American Institute of Physics. [doi:10.1063/1.3597647]

Many natural and technological systems can be described as networks of coupled oscillators. A typical problem in their analysis is to find dynamical features, e.g., synchronization, transition to chaos, etc., in dependence on the properties of oscillators and of the coupling. Here, we address the inverse problem: how to find the properties of the coupling from the observed dynamics of the oscillators. This may be relevant for many experimental situations, where the equations of underlying dynamics are not known, especially for biological applications. We present here a method which is based on invariant reconstruction of phase dynamics equations from multivariate observations, where at least one scalar oscillating observable of each oscillator must be available. The method includes several algorithmic steps which are rather easy to implement numerically. We illustrate the method by numerical examples of small networks of van der Pol oscillators.

I. INTRODUCTION

Dynamics of coupled self-sustained oscillators attracts vast interest of researchers, both due to a variety of nontrivial effects and to numerous applications in physics, chemistry, and life sciences. In modeling, one usually focuses on the relations of the dynamical features, such as synchronization patterns, to the coupling structure of the underlying network. In experimental data analysis, an inverse problem typically arises, where one wants to reveal the interactions from the observed dynamics. A simple consideration demonstrates that this task is really challenging and highly nontrivial: indeed, computation of correlations and/or of a synchronization index between the oscillators does not solve the problem, because the interactions are generally non-reciprocal while both the correlation and the synchronization index are symmetric measures.

There exist several approaches for the determination of directional coupling. The first one exploits various informa-

tion-theoretical techniques and quantifies an interaction with the help of such asymmetric measures as transfer entropy, Granger causality, predictability, etc.^{1–5} The second approach is based on the idea of generalized synchronization and relies on state-space based criteria of mapping of nearest neighbors (see references in Ref. 6). The third approach developed in our previous publications,^{7,8} see also Refs. 9 and 10, is based on an estimation of phases from oscillatory time series and on a subsequent reconstruction of phase dynamics equations and quantification of coupling by inspecting and quantifying the structure of these equations. The disadvantage of the third, dynamical, approach in comparison to the information-theoretical one is that it is restricted to the case of rather weakly coupled oscillatory systems. However, for this widely encountered case, the dynamical approach has clear advantages: it works with phases, which are most sensitive to interaction and it yields description of connectivity which admits a simple interpretation. This consideration is supported by several comparative studies.^{6,12} The dynamical approach has been tested in a physical experiment¹⁶ and used for analysis of connectivity in a network of two or several oscillators in the context of cardio-respiratory interaction,¹⁷ brain activity,^{18–20} and climate dynamics.²¹ Notice that there is an intermediate group of techniques which apply informational measures to time series of phases.^{8,13–15}

Recently, we have essentially improved the dynamical approach by suggesting a technique for an *invariant* reconstruction of the phase dynamics equations from bivariate time series.^{22–24} Invariance in this context means that the reconstructed equations do not depend on the observables used, at least for a wide class of observables. This technique was tested on numerical examples as well as in physical experiments with coupled metronomes^{22,23} and with electrochemical oscillators.²⁵ In this paper, we extend the technique to cover the case of small networks of interacting oscillators. Starting from the scalar time series for each node of the network, we recover the directional connectivity. Note that,

contrary to other papers exploiting phase dynamics approach,^{10,11} we do not assume a closeness of oscillator frequencies or a uniform distribution of their phases. Before proceeding with the presentation of the method, we discuss what kind of connectivity we reconstruct.

Suppose an oscillator k is effected by an oscillator l via a physical connection, e.g., a resistor, an optical fiber, a synapse, etc. Mathematically, this is reflected by an explicit dependence of the time evolution of the state variables of system k on the state variables of system l , cf. Eq. (1) below. We say that these oscillators are directly linked, or structurally connected. If the phase dynamics of oscillator k explicitly depends on the phase of system l , we say that the oscillators are effectively phase connected. The important issue, which to the best of our knowledge has not yet been addressed theoretically, is that the connectivity in terms of the phases differs from that in the original (full) equations. The reason for this is the appearance of additional terms in high-order approximations in the process of a perturbative reduction of full equations to the phase dynamics. As a result, some nodes, which are not directly connected, may appear effectively phase connected. Although we are not able to demonstrate the difference between structural and effective connectivity analytically, we provide below some qualitative arguments and support them by numerics. We explain that our method, like any other method based on the phase dynamics, yields the information on the effective phase connectivity. Next, we note that oscillators k, l can exhibit correlated or synchronized behavior due to a common driving from oscillator m . This correlation can appear even in the absence of any direct connections between k and l (see the discussion of Fig. 3(c) below); this case is sometimes denoted as functional connectivity. We demonstrate that our approach reliably distinguishes between correlation due to a common input and true interaction.

The paper is organized as follows. In Sec. II, we discuss the phase dynamics description of an oscillatory network and the difference between the connectivity in full and phase equations. Next, in Sec. III, we describe the basic method for the phase dynamics reconstruction; it is illustrated in Secs. IV and V. We summarize and discuss our results in Sec. VI.

II. PHASE DESCRIPTION OF OSCILLATOR NETWORK

We assume that an individual oscillator is described by variables \mathbf{x} , which satisfy a system of ordinary differential equations (ODEs) $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, and that this equation system possesses a stable limit cycle. The latter can be parametrized by the phase φ which grows uniformly in time, $\dot{\varphi} = \omega = \text{const}$, where ω is the oscillation frequency. Notably, this phase parametrization is unique up to trivial shifts and is invariant with respect to variable transformations.^{26,27}

A network of N coupled oscillators can be represented as

$$\dot{\mathbf{x}}_k = \mathbf{G}_k(\mathbf{x}_k) + \varepsilon \mathbf{H}_k(\mathbf{x}_1, \mathbf{x}_2, \dots), \quad k = 1, \dots, N, \quad (1)$$

where ε characterizes the coupling strength and the coupling functions \mathbf{H}_k generally depend on the states of all oscillators. Note that generally the oscillators can be quite different, e.g.,

even the dimensions of state variables \mathbf{x}_k can differ. Of course, the functions \mathbf{G}_k can be different as well. If variables \mathbf{x}_l are absent in \mathbf{H}_k , we say that there is no direct coupling from l to k . If the function \mathbf{H}_k can be written as $\mathbf{H}_k = \sum_{j \neq k} \mathbf{H}_{kj}(\mathbf{x}_k, \mathbf{x}_j)$, we call this type of coupling pairwise. Otherwise, if \mathbf{H}_k contains terms, depending on at least two variables acting on k , e.g., $\mathbf{H}_{kjl}(\mathbf{x}_k, \mathbf{x}_j, \mathbf{x}_l)$, we speak about cross-coupling.

If the coupling between the elements of the network is not too strong,²⁸ so that an attracting invariant N -torus exists in the phase space of the full system (1), then the motion on this torus can be parametrized by N phases:

$$\dot{\varphi}_k = \omega_k + h_k(\varphi_1, \varphi_2, \dots), \quad k = 1, \dots, N. \quad (2)$$

The new coupling functions h_k can be approximately obtained in the form of a series

$$h_k(\varphi_1, \varphi_2, \dots) = \varepsilon h_k^{(1)}(\varphi_1, \varphi_2, \dots) + \varepsilon^2 h_k^{(2)}(\varphi_1, \varphi_2, \dots) + \dots, \quad (3)$$

with the help of a perturbative reduction from Eq. (1) to Eq. (2) (see Refs. 26 and 27 for details). However, to the best of our knowledge, a derivation of the second-order and the high-order terms has not yet been performed theoretically. Nevertheless, we can draw some conclusions about the structure of h_k . In the first-order approximation, $h_k^{(1)}$ depends explicitly on phase φ_l only if \mathbf{H}_k depends explicitly on \mathbf{x}_l , i.e., if there exists a direct link $l \rightarrow k$. Thus, if the coupling is pairwise, then in the leading order of approximation only the pairwise terms like $\varepsilon h_{kl}^{(1)}(\varphi_k, \varphi_l)$ appear on the right hand side (RHS) of the system (2). However, in the higher approximations, the RHS of Eq. (2) may contain high-order terms which depend not only on the phases of directly coupled oscillators, but on the other phases as well.²⁹ Thus, generally the phase oscillators Eq. (2) are all-to-all coupled. As a result, the structural and the effective connectivities, i.e., the structures of the functions h_k and \mathbf{H}_k , generally differ. It is interesting to note that such a problem does not appear for two coupled oscillators, where the phase dynamics terms in all orders of approximation in ε have the same structure. In Sec. IV, we support the presented considerations by numerics.

The main idea of our approach is to reconstruct the phase dynamics (2) from multivariate time series; here it is assumed that k -th component of the series represents the output of the k -th oscillator. Next, we decompose the reconstructed coupling functions h_k into several components and, comparing the norms of these components, characterize the partial couplings between particular nodes. In fact, we attribute links only to those nodes for which the corresponding norms are sufficiently large. As discussed above, the structure of the phase network (2), reconstructed in this way, generally differs from the structure of the original one. However, as confirmed by numerical examples below, it represents the original network (1).

We emphasize that validity of the phase equations (2) is not restricted to the range of coupling where the first-order approximation, which can be obtained analytically,²⁶ is

sufficiently good. As long as an attracting invariant torus exists, we can try to reconstruct Eq. (2) from data, although the analytical derivation of these equations is not yet possible.

Before proceeding with the description of the method, we make another remark on the range of its validity. An important dynamical regime in system (2) is that of full or partial synchrony, when the dynamics of the phases reduces to a stable torus of lower dimension (partial synchrony) or to a stable limit cycle (if all oscillators are synchronized). In case of synchrony, our (and similar ones) technique fails because the data points do not cover the original N -dimensional torus and the coupling functions cannot be reconstructed. Sufficiently strong noise or intentional perturbation of the network could cause deviation of the trajectory from the synchronization manifold and help to infer the dynamics, but discussion of this case goes beyond the scope of this paper.

III. RECONSTRUCTION OF THE PHASE DYNAMICS FROM DATA

The main assumption behind our approach is that we have multivariate observations of coupled oscillators, where at least one scalar oscillating time series $y_k(t) = y_k(\mathbf{x}_k(t))$ is available for each oscillator. The first step is to transform this observable into a cyclic observable. Typically, this is done via construction of a two-dimensional embedding (y_k, \bar{y}_k) , where \bar{y} can be, e.g., the Hilbert transform of $y(t)$, see Refs. 23 and 27 for a detailed discussion. Alternatively, one can use for \bar{y} the time derivative of y . In all cases, the data usually require some preprocessing, e.g., filtering. If the trajectory in the plane (y_k, \bar{y}_k) rotates around some center, one can compute a cyclic variable $\theta_i(t)$, e.g., by means of the arctan function. As has been in detail argued in Ref. 23, in this way we obtain not the genuine phase φ_k of the oscillator, which enters Eq. (2), but a cyclic variable, or a protophase. The latter depends on the particular embedding and on the parametrization of rotations and is therefore not invariant. This can be seen already from the analysis of an autonomous oscillator: the described procedure generally yields a protophase $\theta(t)$ which rotates not uniformly, but obeys

$$\dot{\theta} = f(\theta).$$

The transformation to the uniformly rotating genuine phase φ reads

$$\varphi(\theta) = \omega \int_0^\theta \frac{d\theta'}{f(\theta')},$$

where $\omega = 2\pi[\int_0^{2\pi} f^{-1}(\theta')d\theta']^{-1}$ is the frequency of oscillations. For practical reason, it is convenient to determine not the function f , but the probability density of the protophase $\sigma(\theta) = \frac{\omega}{2\pi f(\theta)}$. Thus, the transformation from the protophase θ to the phase φ can be written as

$$\varphi(\theta) = 2\pi \int_0^\theta d\theta' \sigma(\theta').$$

Hence, a determination of the genuine phase from the data reduces to a problem of finding the probability distribution density of the obtained protophase θ , which is a standard task in the statistical data analysis. Practically, one can use either a Fourier representation of the density $\sigma(\theta)$, as in Ref. 23, or a kernel function representation of $\sigma(\theta)$.

After the transformation $\theta_k(t) \rightarrow \varphi_k(t)$ is performed, we reconstruct the phase dynamics making use of the fact that the time derivatives $\dot{\varphi}_k$ are 2π -periodic functions of the phases, in accordance with Eq. (2). These functions can be obtained from the observed multivariate time series of phases $\Phi_k(t)$ with the help of a spectral representation technique²³ or by means of a kernel function estimation. Here, we present the formulas for the spectral technique (see Sec. IV A in Ref. 23 for details), generalized for the case of more than two oscillators:

$$\begin{aligned} F^{(k)}(\varphi_1, \varphi_2, \dots) &= \frac{F_u^{(k)}(\varphi_1, \varphi_2, \dots)}{F_d^{(k)}(\varphi_1, \varphi_2, \dots)}, \\ F_{u,d} &= \sum_{l_1, l_2, \dots} f_{l_1, l_2, \dots}^{(u,d)} \exp(il_1\varphi_1 + il_2\varphi_2 + \dots), \\ f_{l_1, l_2, \dots}^{(u)} &= \frac{1}{T} \int_0^{T} d\Phi_k \exp(il_1\Phi_1 + il_2\Phi_2 + \dots), \\ f_{l_1, l_2, \dots}^{(d)} &= \frac{1}{T} \int_0^T dt \exp(il_1\Phi_1 + il_2\Phi_2 + \dots). \end{aligned} \quad (4)$$

As a result, we obtain the system of the phase dynamics equations in the form

$$\begin{aligned} \frac{d\varphi_k}{dt} &= F^{(k)}(\varphi_1, \varphi_2, \dots) \\ &= \sum_{l_1, l_2, \dots} \mathcal{F}_{l_1, l_2, \dots}^{(k)} \exp(il_1\varphi_1 + il_2\varphi_2 + \dots). \end{aligned} \quad (5)$$

The coupling functions $F^{(k)}$ describe all types of interactions: pairwise, triple-wise, quadruple-wise, and so on. To characterize them separately, we calculate the norms of different coupling terms (the partial norms), as follows. The term $\mathcal{F}_{0,0,\dots}^{(k)}$ is a constant (phase-independent) one, it corresponds to the natural frequency of oscillations. The pairwise action of oscillator j on oscillator k is determined by those components of $F^{(k)}$ which depend on phases φ_k and φ_j only. We quantify this action by the partial norm $\mathcal{N}_{k|j}^{(2)}$; note that here the upper index corresponds to the order of interaction (pairwise here). The partial norm can be computed as

$$[\mathcal{N}_{k|j}^{(2)}]^2 = \sum_{l_k, l_j \neq 0} |\mathcal{F}_{0, \dots, 0, l_k, 0, \dots, 0, l_j, 0, \dots}^{(k)}|^2. \quad (6)$$

Correspondingly, the joint action of oscillators j, m on oscillator k is determined by the cross-coupling terms containing three phases $\varphi_k, \varphi_j, \varphi_m$. This action is quantified by the following partial norm:

$$[\mathcal{N}_{k|jm}^{(3)}]^2 = \sum_{l_k, l_j \neq 0, l_m \neq 0} |\mathcal{F}_{0, \dots, 0, l_k, 0, \dots, 0, l_j, 0, \dots, 0, l_m, 0, \dots}^{(k)}|^2. \quad (7)$$

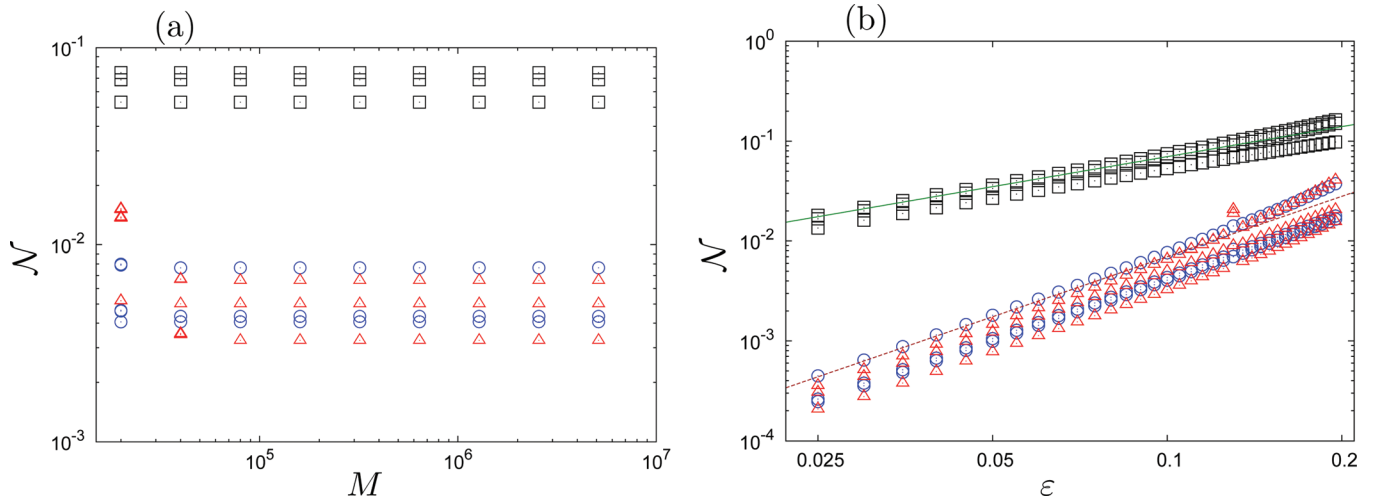


FIG. 1. (Color online) Three van der Pol oscillators in a ring configuration, cf. Fig. 3(a). (a) The partial norms of the pairwise (squares and circles) and cross-couplings (triangles) versus the length M of the used data set. The couplings which exist in Eq. (8) are shown by squares, while the absent links are shown by circles. For each value of M , we plot the results of 20 calculations of the norms for randomly chosen initializations of the system. For all values of $M > 20000$, the results almost perfectly coincide (differences are of the order $\sim 10^{-6}$) so that the markers overlap; for $M = 20000$, some spreading (differences $\sim 10^{-4}$) can be seen. (b) The partial norms of the coupling terms versus the coupling parameter ε , for fixed data length $M = 10^6$. The green solid line has slope one while the dashed brown line has slope two.

Similarly, we can compute the partial norm $\mathcal{N}_{kljmn}^{(4)}$, and so on.

Now, we discuss the terms $\mathcal{F}_{0,\dots,0,l_k,0,\dots,0}^{(k)}$ which depend only on one phase φ_k and, therefore, cannot be attributed to any interaction. In Ref. 23, these terms have been eliminated by means of an additional transformation $\varphi \rightarrow \varphi'$, using some additional assumptions on the structure of coupling. In examples presented below, we checked that the difference between the formulations in terms of φ and φ' is rather small. Therefore, we use the technique described above and neglect the small terms $\mathcal{F}_{0,\dots,0,l_k,0,\dots,0}^{(k)}$.

IV. CASE STUDY: THREE COUPLED VAN DER POL OSCILLATORS

In this section, we test the presented technique on a system of three coupled van der Pol oscillators:

$$\begin{aligned} \ddot{x}_1 - \mu(1 - x_1^2)\dot{x}_1 + \omega_1^2 x_1 &= \varepsilon[\sigma_{12}(\dot{x}_2 + x_2) + \sigma_{13}(\dot{x}_3 + x_3)] \\ &\quad + \sigma_{11}\eta x_2 x_3, \\ \ddot{x}_2 - \mu(1 - x_2^2)\dot{x}_2 + \omega_2^2 x_2 &= \varepsilon[\sigma_{21}(\dot{x}_1 + x_1) + \sigma_{23}(\dot{x}_3 + x_3)] \\ &\quad + \sigma_{22}\eta x_1 x_3, \\ \ddot{x}_3 - \mu(1 - x_3^2)\dot{x}_3 + \omega_3^2 x_3 &= \varepsilon[\sigma_{31}(\dot{x}_1 + x_1) + \sigma_{32}(\dot{x}_2 + x_2)] \\ &\quad + \sigma_{33}\eta x_1 x_2. \end{aligned} \quad (8)$$

Here, the matrix σ_{kl} , with entries zero and one, determines the coupling structure (topology) of the network: for non-diagonal terms, $\sigma_{kl} = 1$, if there is forcing from oscillator l to oscillator k , and $\sigma_{kl} = 0$, otherwise. Diagonal terms σ_{kk} determine the presence or absence of cross-coupling. Parameters ε and η describe the intensity of the pairwise and of the cross-coupling, respectively. In all examples of this section, we use the values of parameters $\mu = 0.5$, $\omega_1 = 1$, $\omega_2 = 1.3247$, and $\omega_3 = 1.75483$. Equation (8) has been solved by the Runge-Kutta method, and the time series of

x_k, \dot{x}_k were used to calculate the protophases as $\theta_k = -\arctan(\dot{x}_k, x_k)$.

A. Test examples

Before discussing different types of networks, we present a detailed analysis of two particular cases in order to exemplify the appearing statistical and systematic errors. Our second goal is to demonstrate numerically the difference between the structural and the effective connectivities.

We begin by testing the model (8) with unidirectional pairwise couplings, setting $\sigma_{12} = \sigma_{23} = \sigma_{31} = 1$ while $\sigma_{13} = \sigma_{21} = \sigma_{32} = \eta = 0$, cf. Fig. 3(a). First, we analyze statistical effects of the number of data points used for the analysis. To this end, we fix $\varepsilon = 0.1$ and vary the length M of the data set. The data points are sampled with the time step 0.05. The results for the partial norms of the reconstructed coupling, presented in Fig. 1(a), demonstrate that there is almost no statistical spreading.

In another test (Fig. 1(b)), we fix the data set size $M = 10^6$ and reconstruct the norms of the coupling terms in dependence on the coupling parameter ε . One can see that while the couplings existing in model (8) scale as $\sim \varepsilon$, the other coupling terms, including the cross-coupling, scale as $\sim \varepsilon^2$. This confirms the qualitative consideration in Sec. II above, where we discussed the difference between the structural and the effective connectivities. There we argued, that in addition to the terms which appear in the first-order approximation of the phase dynamics and which correspond to the links, existing in the original system, new terms appear in the second and higher orders. Unfortunately, the differentiation between these high-order terms and spurious terms due to a possible systematic bias of the method (cf. Fig. 2 below) is not possible unless the theoretical phase description which includes high-order terms is available.

Next, we consider a configuration, where oscillator 2 drives unidirectionally oscillators 1 and 3, cf. Fig. 3(e).

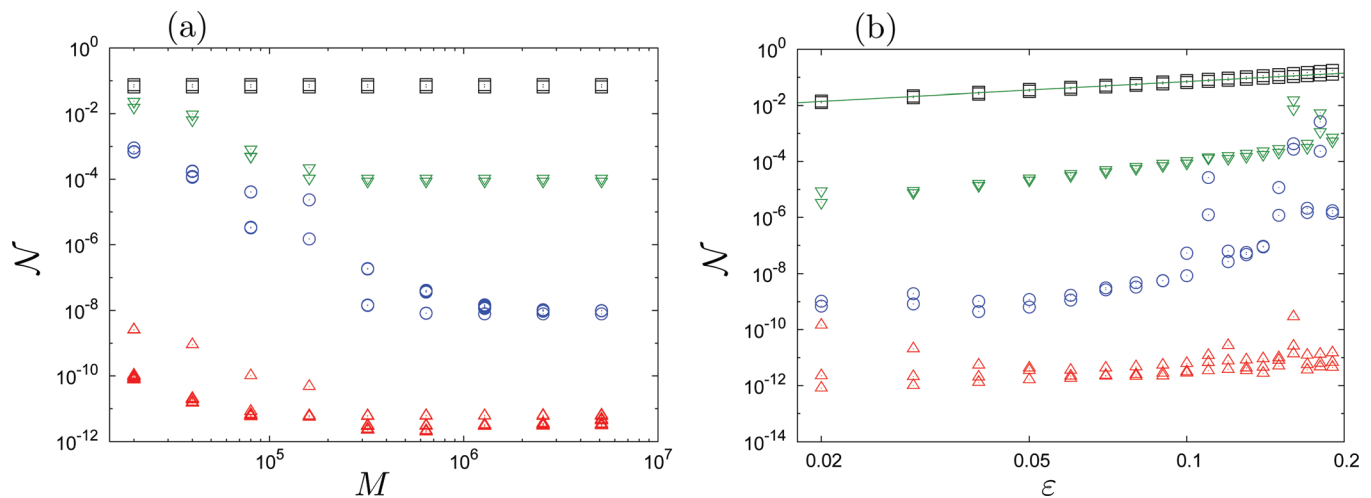


FIG. 2. (Color online) Three van der Pol oscillators in a configuration, where oscillator 2 drives oscillators 1 and 3, cf. Fig. 3(e). The partial norms of the coupling terms are plotted for fixed coupling strength $\varepsilon = 0.1$ vs the number of points M (a) and for fixed $M = 10^6$ vs ε (b). Squares correspond to truly existing links. Triangles up represent all terms of the 2nd coupling function; these terms should be zero. Triangles down correspond to triple terms $2,3 \rightarrow 1$ and $1,2 \rightarrow 3$ and circles correspond to pairwise terms $1 \rightarrow 3$ and $3 \rightarrow 1$; these terms are spurious as well. It is seen that almost for the whole range of ε , the true terms are at least 10^3 times larger than spurious ones. The green solid line in (b) has slope one.

Again, we first compute the partial norms of the coupling terms for fixed $\varepsilon = 0.1$ and varied M and then for fixed $M = 10^6$ and varied ε . The results for the partial norms of the reconstructed coupling are given in Figs. 2(a) and 2(b), respectively. For this configuration, it is obvious that there should be only two nonzero terms, reflecting the pairwise action $2 \rightarrow 1$ and $2 \rightarrow 3$. The results show that the separation between the true and spurious terms remains good (at least 2 orders of magnitude) even for a relatively strong coupling. Fig. 2(a) shows that the spurious terms are due to systematic errors, as for the data sets with $M > 10^6$ their fluctuations are very small. Mostly small (of order 10^{-11} , close to round-off errors) are the terms describing the dynamics of the non-driven oscillator. False pairwise and cross-coupling terms have norms $\sim 10^{-8}$ and 10^{-4} , respectively. We partially attribute the appearance of such terms to the small correlations between the phases that appear due to couplings: e.g., as a result of a correlation between φ_3 and φ_2 , the dynamics of φ_1 may be slightly better described when the values of φ_3 are included. While postponing a more detailed analysis of the spurious terms to the future, we would like to emphasize that these terms are almost two order of magnitude smaller than the combinational coupling terms of Fig. 1(a), this allows us to assume that the terms in Fig. 1(a) are true and not spurious.

B. General networks of three oscillators

As demonstrated above, for a sufficiently large data set, the reconstruction method yields phase couplings almost without statistical errors. It reproduces both the coupling terms which exist in the original Eq. (1), as well as the additional terms appearing in the phase model (2). Here, we present the results for different types of coupling of three van der Pol oscillators (8). The data sets consisted of 10^6 points sampled with 0.01 time step. The analyzed coupling configurations are schematically given in the left column of

Figs. 3 and 4. Note that we chose the oscillator frequencies and parameters of coupling so that the network remains asynchronous.

In the first set of tests, we considered only pairwise coupling in Eq. (8), i.e., we took $\sigma_{kk} = 0$. The results for $\varepsilon = 0.05$ and $\varepsilon = 0.15$ are presented in the middle and right columns of Figs. 3 and 4, respectively. The numerical values of all reconstructed norms can be seen in the tables in the middle column. Here, we show the coupling norms for the pairwise coupling (6) as non-diagonal terms, and the triple-phase coupling according to Eq. (7) as diagonal ones. The coupling terms, which are present in the original system (8) (i.e., those with $\sigma_{kl} = 1$) are shown in boxes. We see that in all cases, these terms are definitely larger than other entries of the reconstructed coupling matrix, i.e., the reconstruction works pretty well in all cases. Comparing the results for smaller and larger coupling (Figs. 3 and 4, respectively), we see that a larger coupling strength leads to an appearance of cross-coupling terms for phases that are not present in the original system (8).

In the right columns of Figs. 3 and 4, we show schematically the reconstructed coupling configurations. The corresponding partial norms are coded by the width of arrows which link the nodes. Only norms which are larger than 10% of the maximal norm are shown here. This cut-off threshold was chosen rather arbitrary in order to ensure clearness of the visual representation of the results, given in the corresponding tables; this choice is supported by the results of the case study, presented in Figs. 1 and 2. We emphasize, that due to the difference between structural and effective connectivities and to the absence of a possibility to derive the exact phase model theoretically, it appears hardly possible to give a clear recipe for a separation between the existing and the spurious couplings. One possibility would be an analysis of coupling strength dependence of different terms, like in Fig. 1, but for a given observation with some fixed coupling strength this appears hardly possible.

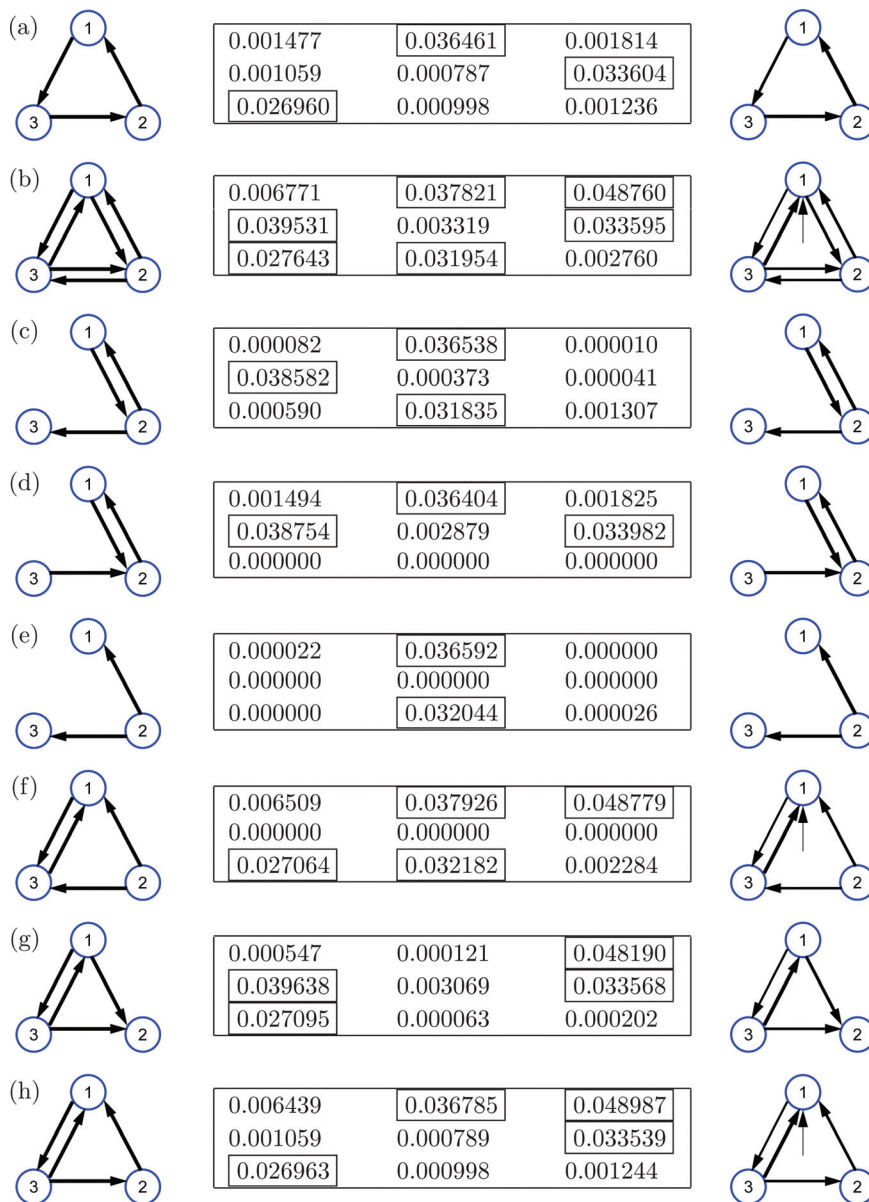


FIG. 3. (Color online) True configurations in the full model (8) (left column), to be compared with the configurations of the reconstructed phase model. Each panel in the middle column shows the partial norms of the reconstructed coupling functions as a table; the numbers in boxes correspond to truly existing links and the numbers without the boxes differ from zero either due to the difference between structural and effective connectivities or due to numerical artifacts. Schematic pictures in the right column present the largest norms only (see text); the values of the norms are coded by widths of linking arrows. The arrows from the center to the first oscillator (see panels (b), (f), and (h)) reflect the joint action from oscillators 2 and 3 (cross-coupling term) which cannot be decomposed into pairwise actions $2 \rightarrow 1$ and $3 \rightarrow 1$. Coupling strengths in the model are $\varepsilon = 0.05$ and $\eta = 0$.

In the second set of tests, we included the cross-couplings by setting $\sigma_{kk} = 1$ and $\eta = 0.1$; the results are presented in Fig. 5. Here, each configuration shown in Figs. 3(a)–3(h) was complemented by the links according to Fig. 5(i). We see that basically the coupling structure is correctly reproduced by the method, although due to a joint action of different terms some couplings, not presented in the original system (8), do not fall below 10% of the largest norm.

Summarizing the results of this section, we conclude that the connectivity of a weakly and pairwise coupled network can be correctly revealed. For stronger coupling, all really existing in the original network links are correctly revealed and some additional links appear. However, the latter are not necessarily spurious. Indeed, as discussed above, not very weak pairwise coupling in the original network may lead to additional pairwise connections in the network of phase oscillators and to cross-coupling terms. Our results are in full agreement with this argument.

V. CASE STUDY: NETWORKS OF FIVE AND NINE OSCILLATORS

In this section, we report on the results for small networks of $N = 5$ and $N = 9$ coupled van der Pol oscillators. We simulated the systems with pairwise interaction only,

$$\ddot{x}_k - \mu(1 - x_k^2)\dot{x}_k + \omega_k^2 x_k = \varepsilon \sum_{l \neq k} \sigma_{kl} (x_l \cos \alpha_{kl} + \dot{x}_l \sin \alpha_{kl}). \quad (9)$$

As above, $\mu = 0.5$, ε is the intensity of coupling, and σ_{kl} is the $N \times N$ connectivity matrix, i.e., $\sigma_{kl} = 1$ if oscillator l acts on oscillator k , and $\sigma_{kl} = 0$ otherwise. An additional parameter α_{kl} describes the phase shift in the coupling, so that generally the coupling can be either attracting or repelling.

We note that the computational efforts and the data requirements for reconstruction of a network of N oscillators grow rapidly with N , so that full reconstruction for

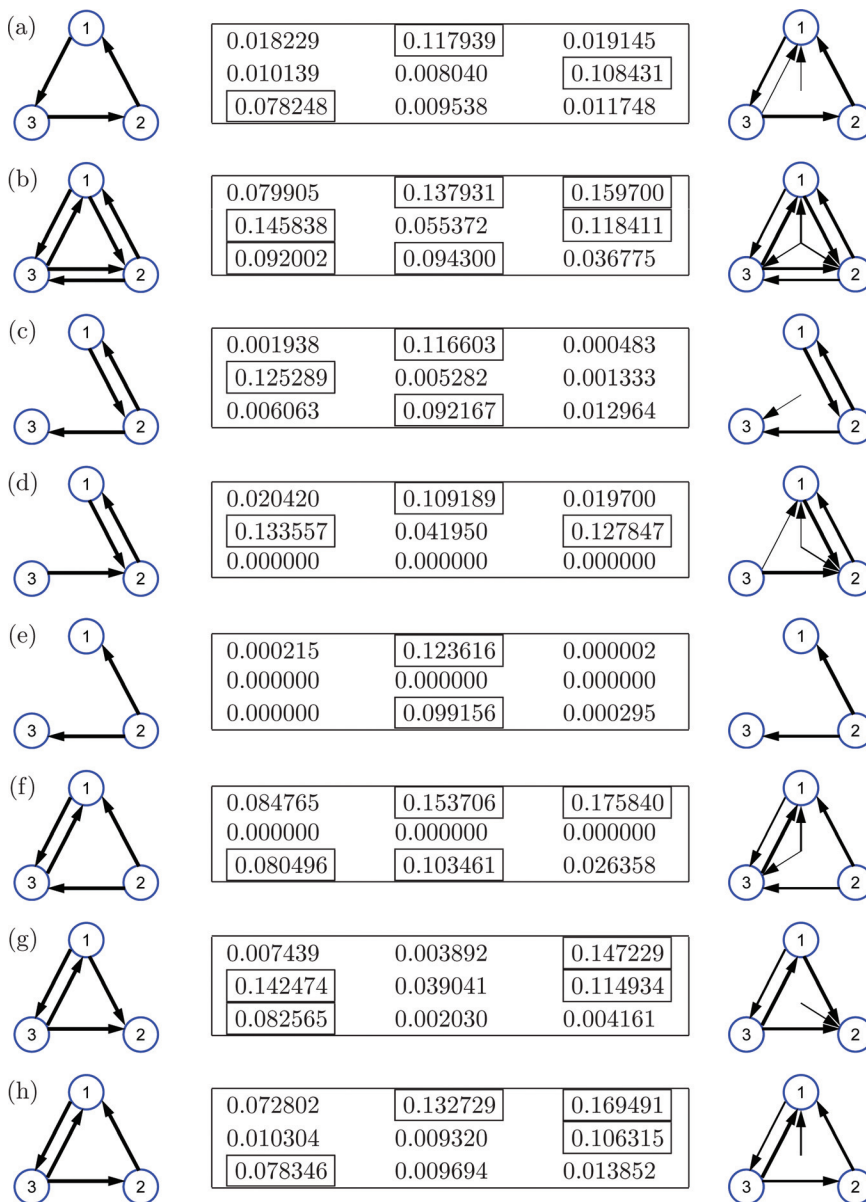


FIG. 4. (Color online) Same as Fig. 3, but for $\varepsilon = 0.15$. Again, an arrow from the center to, say, first oscillator reflects the joint action from oscillators 2 and 3.

$N > 3$, though theoretically possible, see Eqs. 4 and 5, becomes practically unfeasible. Indeed, to reconstruct all the possible coupling terms, one needs enough data points to fill the N -dimensional cube $0 \leq \varphi_k < 2\pi$. Our tests demonstrate that one needs about 100^N data points (cf. Fig. 2(a) where for a reliable reconstruction for three oscillators some 10^6 data points are needed). Additionally, the number of terms to be reconstructed also grows as const^N . Both these factors result in a growth of CPU time by a factor ~ 1000 when the number of oscillators increases by one (the computation time for $N = 3$ is of the order of several minutes on a workstation). On the other hand, in Sec. IV, we have verified that weak pairwise couplings can be reliably recovered. Therefore, for networks described by Eq. (9), we reconstruct pairwise coupling functions only, for all pairs of network units.

We performed a statistical analysis of the model (9): for many randomly chosen values of frequencies ω_k , parameters

α_{kl} , and connectivity matrices σ_{kl} , we analyzed pairwise interactions in the networks. For $N = 5$, the frequencies ω_k have been chosen uniformly distributed in $(0.5, 1.5)$, and the number of incoming links for each node was 2, i.e., $\sum \sigma_{kl} = 2$. The links have been chosen randomly; the incoming and the outgoing links were chosen independently. For $N = 9$, the frequencies ω were distributed uniformly in $(0.5, 2.5)$, and the number of incoming links was 3. The intensity of coupling in both cases was $\varepsilon = 0.15$, the phase shift α_{kl} was distributed uniformly in $(0, 2\pi)$.

For each network, we computed N protophases according to $\theta_k = -\arctan(\dot{x}_k, \omega_k x_k)$ and then transformed them to phases φ_k . Next, a synchronization analysis has been performed: for all pairs, we calculated the synchronization index $|\langle e^{i(\varphi_k - \varphi_l)} \rangle|$. As our technique does not work in case of synchrony, we excluded from the further consideration all networks where the synchronization index was larger than 0.5 at least for one link.



FIG. 5. (Color online) Same as Fig. 3, but for combination of pairwise and triple-coupling, $\varepsilon = \eta = 0.1$. In the left column, an arrow from the center to, say, first oscillator reflects the term $\sim x_2 x_3$ with $\sigma_{11} = 1$ in Eq. (8), etc.

For each obtained non-synchronous network, we fitted the time derivatives $\dot{\varphi}_k$ by $\sum_{k \neq l} F_{kl}(\varphi_k, \varphi_l)$, where F_{kl} depend on two phases only. Next, we computed the norms \mathcal{N}_{kl} of all F_{kl} ; these norms represent the reconstructed connectivity of the network. Performing the analysis with a large ensemble of non-synchronous networks, we separated \mathcal{N}_{kl} into two classes: the first class contains those values of \mathcal{N}_{kl} for which $\sigma_{kl} = 1$, i.e., the connections between the nodes k, l in the network (9) really exists. Otherwise, if $\sigma_{kl} = 0$, the value \mathcal{N}_{kl} belongs to the second class. The results are summarized in Fig. 6. Ideally, one would expect that all norms in the first class are much larger than those in the second class (cf. Figs. 3 and 4). We see that there is a clear, although not

ideal, separation between the connected and the non-connected links, what allows us to conclude that generally our technique correctly reproduces the structural connectivity of the network. This picture is in agreement with the results for three coupled oscillators, where we never obtained small norms for the true links, while sometimes large pairwise links have been detected that not existed in the original equations. We expect that also for the regimes shown in Fig. 6, the high values of the norms for absent original connections are mainly not spurious but reflect the effective phase connectivity. However, a more detailed statistical analysis of these issues is needed before a “blind” application of the method would be possible.

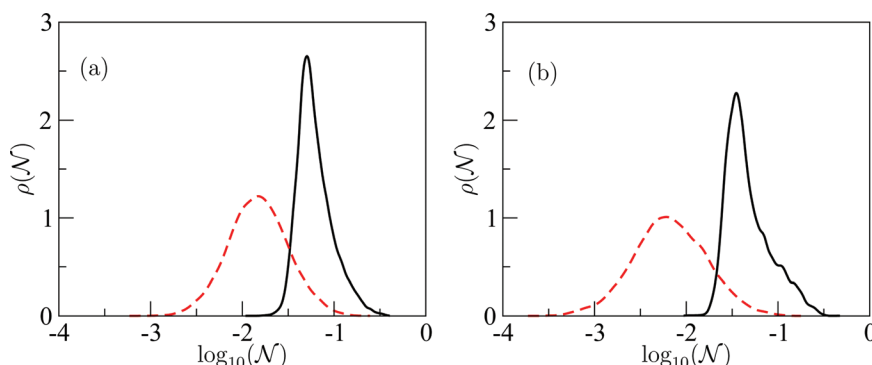


FIG. 6. (Color online) Probability distribution densities $\rho(N)$ of the norms of reconstructed couplings: black solid line: really present connections ($\sigma_{kl} = 1$), dashed red line: absent connections ($\sigma_{kl} = 0$), for networks (9) of 5 (a) and 9 (b) oscillators.

VI. DISCUSSION AND CONCLUSION

In summary, we have presented a technique for the reconstruction of the phase model of a network of coupled limit cycle oscillators. From the reconstructed model, we infer directional couplings of the network. We have tested the technique on small networks of van der Pol oscillators with pairwise and cross-coupling. We have shown that the presence and direction of existing links can be reliably revealed. However, as argued in Sec. II, the effective phase connectivity generally differs from the structural connectivity: if matrix σ_{kl} determines the structural connectivity of the original network model (1), then the phase model (2) for this network would have an effective connectivity matrix Σ_{kl} with a larger number of links and possibly with many cross-couplings. It is this effective connectivity what is determined in our approach, not the structural one. Therefore, our technique generally yields additional links which are absent in the original network. These links are not spurious, but correspond to the higher-order terms in the phase dynamics; they become more pronounced with increase of the coupling. On the other hand, some additional links may be artifacts of the method (spurious links). The source of these artifacts requires a further examination. In particular, in our approach we completely ignored the variations of the amplitudes of the reconstructed signals, what can be a source of errors. Artifacts may appear also due to noise, closeness to synchrony, etc., so that the differentiation between the truly existing and the spurious links is not perfect. These effects, as well as an analysis of other types of oscillators (including noisy and chaotic ones) are currently under investigation. Nevertheless, our approach allows us to distinguish between the structural and effective connectivities, described by matrices σ , Σ and reflecting true interactions, and the functional one. The term functional connectivity is typically used in the context of the analysis of brain activity and is widely understood as a correlated time behavior; it is quantified by different measures of correlations or synchronization. The functional connectivity results from the dynamics, and may only loosely correspond to the structural and effective ones.^{30,31}

Finally, we discuss the perspectives of the analysis of networks of many oscillators. As we have demonstrated, generally a pairwise analysis yields good results only if the connections are pairwise and weak. In the case of

cross-coupling (see e.g., Fig. 5(i)), pairwise analysis does not help. Determination of applicability of the pairwise analysis for a given network remains an open problem. A possible solution might be a reconstruction of the phase dynamics for several or all triplets of oscillators and comparison of the terms, dependent on three phases, with those dependent only on two. If the triple terms are much smaller in norm than pairwise ones, then the pairwise analysis may be sufficient.

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