

Dynamics of globally coupled oscillators: Progress and perspectives

Arkady Pikovsky and Michael Rosenblum

Institute of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, 14476 Potsdam-Golm, Germany

(Received 10 February 2015; accepted 4 June 2015; published online 1 July 2015)

In this paper, we discuss recent progress in research of ensembles of mean field coupled oscillators. Without an ambition to present a comprehensive review, we outline most interesting from our viewpoint results and surprises, as well as interrelations between different approaches. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4922971>]

Studies of large systems of interacting oscillatory elements are a popular and extensively developing branch of nonlinear science. The number of publications on the subject grows rapidly, with many crucial contributions published in the *Chaos* journal. In addition to purely academic interest, this research finds promising applications in various fields. Representative examples are understanding of pedestrian synchrony on footbridges and of other social phenomena, development of efficient high-frequency power sources, modeling and control of neuronal rhythms, etc. In this paper, we present our view of the recent development of this highly interesting field.

I. INTRODUCTION

In the second half of the 17th century, Kaempfer, Dutch physician, made a journey to South-East Asia and later published a book, describing his trip.¹ In particular, in his memoir, he gives an account of a spectacular show, which happens if a swarm of fireflies occupies a tree: the insects “hide their Lights all at once, and a moment after make it appear again with the utmost regularity and exactness.” This phenomenon of *self-synchronization in a large population of interacting oscillatory objects* not only remains an appealing entertainment—be it an excursion on a night river in Thailand to observe fireflies or cycling in a large group of people, equipped with electronic “bikeflies,” as a part of a festival in Chicago²—but it stays in the focus of scientific interest within many decades. Studies of various aspects of the collective dynamics in large oscillatory networks attract attention of physicists and applied mathematicians, and find applications ranging from electrochemistry to quantum electronics, and from bridge engineering to social sciences.

A particularly popular and expanding field of applications is neuroscience. For the first time, the link between synchronous flashing of the fireflies and origin of the brain rhythms was established, on a qualitative level, by the famous mathematician Wiener in his monograph on *Cybernetics*,³ in the chapter entitled “Brain waves and self-organizing systems,” see also Ref. 4. Putting forward the hypothesis that the brain waves emerge due to “phenomenon of the pulling together of frequencies,” he questioned, whether the same nonlinear mechanism takes place in case of fireflies, crickets, and other species exhibiting collective

oscillatory behavior, and suggested that experiments on fireflies and on electronic systems can shed light on the brain wave dynamics.

Since that many experiments have been reported, including those with electrochemical^{5,6} and electronic oscillators,⁷ metronomes,⁸ Josephson junctions,⁹ laser arrays,¹⁰ yeast cells,¹¹ and gene-manipulated clocks in bacteria.¹² They are complemented by observations of many social phenomena like pedestrian synchrony on footbridges,¹³ synchronous hand clapping in opera houses,¹⁴ egg-laying in bird colonies,¹⁵ and menstrual synchrony¹⁶ (the latter effect remains controversial¹⁷). This research was also accompanied by an essential progress in the theoretical description, which we outline below, along with open questions.

II. SOLVABLE MODELS

A few years after publication of the Wiener’s book, Winfree presented a first mathematical description of collective synchrony in a large population of biological oscillators.¹⁸ Reducing the dynamics of each oscillator to that of only one variable, the phase φ (we will discuss conditions for this reduction below), he proposed the mean-field model

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \Gamma(\varphi_k) \sum_{j=1}^N I(\varphi_j), \quad (1)$$

where $N \gg 1$ is the number of units, ω_k are their natural frequencies, and ε quantifies the strength of the interaction. The function $\Gamma(\varphi_k)$ describes the phase sensitivity of the oscillator to an infinitesimal perturbation and is typically called the phase response curve, PRC. Notice that the PRC can be experimentally obtained by repeated stimulation of an isolated system.¹⁹ Finally, the forcing function $I(\varphi_j)$ describes the effect of the j th unit on the unit k . In this setup, the coupling is assumed to be global, i.e., of the all-to-all type, and functions Γ , I are assumed to be identical for all interactions. Thus, the inhomogeneity of the system is due to a distribution of frequencies ω_k only. Although the model is quite complicated for the analysis, Winfree has shown that it exhibits a transition to a macroscopic synchronized state, characterized by non-zero mean field $N^{-1} \sum_j I(\varphi_j)$. He discovered that the collective synchrony is a threshold phenomenon: the transition occurs if the coupling strength ε is large

enough or the inhomogeneity, i.e., the width of the distribution of ω_k , is sufficiently small. For recent studies on the Winfree model, see Refs. 20 and 21, it has been shown that it can be treated analytically at least if function Γ contains only one Fourier harmonic.

The next pioneering step has been done by Kuramoto in his seminal publication 40 years ago.²² The model he suggested can be considered as the weak-coupling limit of Eq. (1). Indeed, if $\varepsilon \ll \omega$, then each Eq. (1) can be averaged over the oscillation period; then each term $\Gamma(\varphi_k)I(\varphi_j)$ yields a function of the phase difference, $g(\varphi_j - \varphi_k)$, so that the averaged model reads

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^N g(\varphi_j - \varphi_k). \tag{2}$$

A general case of an arbitrary 2π -periodic function g is discussed later below, while the simplest case $g(\cdot) = \sin(\cdot)$ corresponds to the famous Kuramoto model (notice that the choice of the sine function is not a result of some approximation but just the simplest solvable case)

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_k) = \omega_k + \varepsilon R \sin(\Theta - \varphi_k). \tag{3}$$

Here, $Re^{i\Theta} = N^{-1} \sum_j e^{i\varphi_j}$ is the complex mean field, R and Θ are its amplitude and phase, respectively. Kuramoto solved the problem in the thermodynamic limit $N \rightarrow \infty$, using a self-consistent approach: assuming a harmonic mean field with unknown amplitude and frequency, he obtained closed integral equations for these two quantities. The celebrated result is the existence of the critical coupling, ε_c , proportional to the width of the frequency distribution. For sub-threshold coupling, the mean field is exactly zero, while for $\varepsilon > \varepsilon_c$, a nontrivial solution with non-zero mean field appears; the amplitude of the field grows as $\sqrt{\varepsilon - \varepsilon_c}$ and its frequency equals the central frequency of the distribution of ω_k (which is assumed to be symmetric and unimodal). Thus, appearance of the collective mode can be treated as a second-order nonequilibrium phase transition. As has been shown in Ref. 23, the Kuramoto model with the uniform frequency distribution exhibits a jump of the order parameter.

III. COLLECTIVE DYNAMICS OF THE KURAMOTO MODEL

The results of Kuramoto gave an enormous impact on the development of the field, with still an increasing number of publications on the subject. The model (3) and its extension due to Sakaguchi and Kuramoto,²⁴ who accounted for a possible phase shift in coupling,

$$\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k + \beta), \tag{4}$$

became a paradigmatic model for the analysis of large oscillator ensembles.

A. Watanabe-Strogatz theory

We now briefly discuss an interesting and important mathematical property of the Kuramoto-Sakaguchi model (4), namely, its *partial integrability*. We start this discussion by consideration of the simplest case of identical units, $\omega_k = \omega$. As has been shown in the seminal publications by Watanabe and Strogatz²⁵ (WS), the dynamics of this system can be described by three global variables ρ , Φ , α and $N - 3$ constants of motion ψ_k . Here, the variable $0 \leq \rho \leq 1$ is roughly similar to the mean field amplitude R ; Φ and α are angular variables; often it is convenient to combine two variables as $z = \rho e^{i\Phi}$, see also Ref. 26. The main idea of the powerful WS theory is as follows. Consider $N > 3$ identical oscillators governed by

$$\dot{\varphi}_k = \omega(t) + \text{Im}[H(t)e^{-i\varphi_k}], \tag{5}$$

where $H(t)$ is an arbitrary common forcing. Obviously, Eq. (4) is a particular case of Eq. (5). The latter can be re-written as

$$\frac{d}{dt}(e^{i\varphi_k}) = i\omega(t)e^{i\varphi_k} + \frac{1}{2}H(t) - \frac{e^{i2\varphi_k}}{2}H^*(t). \tag{6}$$

Next, we transform N variables φ_k to complex z , $|z| < 1$, and N real ζ_k , according to

$$e^{i\varphi_k} = \frac{z + e^{i\zeta_k}}{1 + z^* e^{i\zeta_k}}. \tag{7}$$

This transformation, found by WS and written in this form in Refs. 26 and 27, is known as the Möbius transformation. Since the system is under-determined, one requires

$$N^{-1} \sum_{k=1}^N e^{i\zeta_k} = \langle e^{i\zeta_k} \rangle = 0. \tag{8}$$

Substituting Eq. (7) into Eq. (6) we obtain, after straightforward manipulations

$$\begin{aligned} \dot{z} + [\dot{z}z^* - z\dot{z}^* + i\dot{\zeta}_k(1 - |z|^2)]e^{i\zeta_k} \\ = i\omega z + \frac{H}{2} - \frac{H^*}{2}z^2 + [i\omega(1 + |z|^2) + (z^*H - zH^*)]e^{i\zeta_k} \\ + \left[\dot{z}^* + i\omega z^* + \frac{H}{2}z^{*2} - \frac{H^*}{2} \right] e^{2i\zeta_k}. \end{aligned} \tag{9}$$

Averaging these equations over k , using Eq. (8) and $\langle \dot{\zeta}_k e^{i\zeta_k} \rangle = 0$, we obtain

$$\dot{z} - i\omega z - \frac{H}{2} + \frac{H^*}{2}z^2 = \left[\dot{z}^* + i\omega z^* - \frac{H^*}{2} + \frac{H}{2}z^{*2} \right] \langle e^{2i\zeta_k} \rangle,$$

and this equation is obviously satisfied, if

$$\dot{z} = i\omega z + \frac{H}{2} - \frac{H^*}{2}z^2. \tag{10}$$

Substitution of Eq. (10) and its complex conjugate into Eq. (9) yields

$$i\dot{\zeta}_k(1 - |z|^2) = \left[i\omega + \frac{z^*H - zH^*}{2} \right] (1 - |z|^2).$$

Excluding the fully synchronous case $|z| = 1$, we obtain $\dot{\zeta}_k = \omega + \text{Im}(z^*H)$. Since the right hand side is independent of k , variables ζ_k differ only by constants. Hence, introducing $\zeta_k = \alpha + \psi_k$, where all $\psi_k = \text{const}$, we finally obtain

$$\dot{\alpha} = \omega + \text{Im}(z^*H). \tag{11}$$

Expressions (10) and (11) represent the Watanabe-Strogatz equations, which completely describe the evolution of an ensemble of identical oscillators.

Extension of the WS theory for the case of non-identical units depends on the structure of the ensemble. Consider first the case of a *hierarchical population*, which consists of M groups of identical units, so that units in each group are subject to the same field.²⁶ In this case, the dynamics of the ensemble obeys the system of M coupled WS equations (10) and (11). Another practically important case is a large population, $N \rightarrow \infty$, which can be characterized by a distribution of frequencies $g(\omega)$. This system is described by WS variables $z(\omega, t)$, $\alpha(\omega, t)$, and the equations for $\partial_t z$, $\partial_t \alpha$ are similar to (10) and (11), see Ref. 26.

B. From WS to Ott-Antonsen theory

There exists a particular case, when the WS equations can be essentially simplified. As has been shown in Ref. 26, for large N and for the uniform distribution of constants of motion ψ_k , the WS variable z coincides with the *local* Kuramoto mean field

$$Z(\omega, t) = \int_0^{2\pi} w(\varphi, t|\omega) e^{i\varphi} d\varphi, \tag{12}$$

(where $w(\varphi, t|\omega)$ is the distribution density of oscillators' phases at frequency ω ; $\int d\varphi w(\varphi, t|\omega) = 1$) to be distinguished from the global mean field

$$Y(t) = \int_{-\infty}^{\infty} g(\omega) Z(\omega, t) d\omega. \tag{13}$$

Substituting $z = Z$ into Eqs. (10) and (11), one obtains closed equations for Z

$$\frac{\partial Z(\omega, t)}{\partial t} = i\omega Z + \frac{H(\omega, t)}{2} - \frac{H^*(\omega, t)}{2} Z^2, \tag{14}$$

$$\frac{\partial \alpha(\omega, t)}{\partial t} = \omega + \text{Im}[Z^* H(\omega, t)]. \tag{15}$$

In the most common case of the mean field coupling $H = \varepsilon e^{i\beta} Y$, the forcing H is independent on α and Eq. (15) becomes irrelevant. Hence, we are left with Eq. (14) which also appears in the recent theory by Ott and Antonsen (OA),^{28,29} briefly introduced below.

Consider the thermodynamic limit of the model

$$\dot{\phi}_k = \omega_k + \text{Im}[H(t)e^{-i\phi_k}] \tag{16}$$

and the corresponding continuity equation for the distribution of phases

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi}(w\dot{\varphi}) = 0. \tag{17}$$

Next, let us introduce Fourier components of the density (the generalized local Daido order parameters)

$$Z_m(\omega, t) = \int_0^{2\pi} w(\omega, \varphi, t) e^{im\varphi} d\varphi, \tag{18}$$

cf. Eq. (12). Computing

$$\frac{\partial Z(\omega, t)}{\partial t} = \int_0^{2\pi} \frac{\partial w}{\partial t} e^{im\varphi} d\varphi = - \int_0^{2\pi} \frac{\partial(w\dot{\varphi})}{\partial \varphi} e^{im\varphi} d\varphi,$$

integrating by parts and using Eq. (16), one obtains an infinite-dimensional system of ordinary differential equations

$$\frac{\partial Z(\omega, t)}{\partial t} = i\omega Z_m + \frac{m}{2}(HZ_{m-1} - H^*Z_{m+1}). \tag{19}$$

A particular case $Z_m = Z_1^m = Z^m$, called the OA manifold, reduces all the equations (19) to a single Eq. (14). Thus, the OA manifold corresponds to the special solution of the WS theory, with the uniform distribution of constants of motion ψ . Furthermore, OA argued that for a continuous frequency distribution $g(\omega)$ the OA manifold is the only attractor²⁹ (although relaxation to it may be rather slow³⁰). A particular case of the Lorentzian distribution $g(\omega) = [\pi(\omega^2 + 1)]^{-1}$ admits a further essential simplification. Under assumption that $Z(\omega)$ is analytic in the upper half-plane, one computes the integral in Eq. (13) by virtue of residues and obtains $Y = Z(i)$. Substituting this along with $\omega = i$ into Eq. (14), one obtains the OA equation

$$\dot{Y} = \left(\frac{\varepsilon e^{i\beta}}{2} - 1 \right) Y - \frac{\varepsilon e^{-i\beta}}{2} Y^2 Y^* \tag{20}$$

for the evolution of the mean field in the Kuramoto-Sakaguchi model. Looking for a stationary synchronous solution, we set $Y = R_0 e^{i\nu t}$ and obtain the amplitude $R_0 = \sqrt{1 - 2/\varepsilon \cos \beta}$ and frequency $\nu = (\varepsilon \cos \beta - 1) \tan \beta$ of the mean field.

For an illustration of the WS and OA theories, we consider the model of two interacting populations by Abrams *et al.*³¹ Both populations consist of the same number of identical oscillators, but the coupling strength within the group differs from that between the groups. The dynamics can be fully described by two coupled systems of WS equations and is therefore six-dimensional.²⁶ In the particular case of uniformly distributed constants ψ , i.e., on the OA manifold, the dimension reduces to four. The latter case, studied in Ref. 31, reveals, for certain parameters, an interesting solution, when one population synchronizes while the other stays between complete synchrony and full asynchrony, i.e., $R_1 = 1$ and $0 < R_2 < 1$, where $R_{1,2}$ are the order parameters of the first and second population, respectively. This symmetry-breaking state is called *chimera*. (Notice that

originally chimera states have been introduced for nonlocally coupled oscillator chains.³²) The order parameter R_2 can be time-periodic. Analysis of the full six-dimensional systems exhibits additional solutions with the quasiperiodic chimera states.²⁶ Theoretical predictions have been confirmed in recent experiments with two groups of metronomes, placed on platforms which were coupled via springs.⁸

C. Generalizations of the Kuramoto model

There are many studies of different generalizations of the Kuramoto model. Here, we briefly mention those where the coupling function is purely harmonic like in Eq. (4), but the overall setup is more complex:

- (a) *Multi-modal frequency distribution and several interacting populations.* We have seen that an ensemble with a Lorentzian distribution of frequencies is described by the OA Eq. (20). A multi-modal distribution of frequencies can be modeled as a superposition of Lorentzians and, hence, described by a system of coupled OA Eq. (20), see, e.g., Ref. 33. Moreover, this approach can be generalized to a set of populations with frequencies, distributed around completely different central values, whereas the latter can be either in resonance³⁴ or not.³⁵
- (b) *Nontrivial transitions for unimodal distributions.* For a long time, it has been assumed that in the Kuramoto-Sakaguchi model (4) with different unimodal distributions of frequencies the dynamics of the mean field is qualitatively the same as for the Lorentzian distribution (Eq. (20)). Rather surprisingly, Omelchenko and Wolfrum³⁶ have demonstrated rather complex transition scenarios, including first-order transitions and bistability, for some unimodal distributions.
- (c) *Complex coupling schemes.* The Kuramoto-Sakaguchi model describes a “direct” coupling scheme: the mean field, calculated algebraically from the states of all oscillators, enters the equations for the phases. The coupling scheme can be, however, more complex: the mean field may act on some macroscopic variables that obey a set of generally nonlinear differential equations, and the acting force is a function of these variables. For example, in a description of pedestrian synchrony on a footbridge,³⁷ one describes each pedestrian by an individual phase variable, but one needs also equations for the swinging mode of the bridge. The latter is driven by the field created by all pedestrians and, in its turn, affects their gaits. Similarly, electronic⁷ or electrochemical^{5,6} oscillators can be coupled through the common macroscopic current or voltage, which obeys macroscopic equations describing the coupling circuit. In this way, one also describes synchronization of Josephson junctions³⁸ or spin-torque³⁹ oscillators.
- (d) *Nonhomogeneous populations.* There is another generalization of the standard Kuramoto-Sakaguchi coupling. The latter assumes that all the oscillators make the same contribution to the mean field and that the mean field acts on all oscillators in an equal way. References 40 and 41 generalized this to the situation,

where the global field still can be introduced, but oscillators contribute to it differently, i.e., with different amplitudes and phase shifts, and the field also acts differently on different oscillators. For a physical implementation, one can consider a receiver which collects signals emitted from the oscillators (where the attenuation and the phase shift are due to signal propagation properties), and the governing signal is then transmitted to the oscillators, being also subject to attenuation and phase shift depending on the positions of the units.⁴¹ One variant of an inhomogeneity of the population is when it consists of two groups with different reaction to the mean-field forcing:⁴² some are “conformists” (they follow the force) and some are “contrarians.” (They tend to be in anti-phase with the forcing.) An alternative classification, quite common for neural ensembles, refers not only to the reaction to the mean-field forcing but also to the way the units contribute to it: some oscillators are inhibitors (they contribute negatively to the mean field, thus trying to desynchronize others), while others exert an excitatory action, contributing positively and thus attempting to synchronize the ensemble.^{43,44}

- (e) *Effects of noise.* Independent noisy forces acting on oscillators of a population counteract synchronization. In this sense, noise is a source of disorder, similar to the distribution of natural frequencies. Due to noise, synchronization is a threshold phenomenon already for identical oscillators and the transition occurs at a critical coupling which is proportional to the noise intensity. In the thermodynamic limit such a system is described analytically by a nonlinear Fokker-Planck equation which differs from the continuity equation (17) by a term $\sim \sigma^2 \frac{\partial^2 w}{\partial \varphi^2}$, where σ is the amplitude of the noise. Completely opposite is the action of a common noise: it tends to synchronize the population of oscillators, and for identical units this effect can be described within the WS framework.⁴⁵
- (f) *Finite-size fluctuations.* Kuramoto and OA theories have been developed in the thermodynamic limit of infinitely large populations; WS theory applies to any population size, but for identical populations only. In finite populations with different natural frequencies of units, one expects to observe fluctuations, both prior and beyond the synchronization transition, which is defined ambiguously in this case. In Ref. 46, the Kuramoto model with a uniform distribution of frequencies and a relatively small number of oscillators have been shown to be chaotic prior to synchronization transition, the maximal Lyapunov exponent however decreases with the system size as $\lambda \sim N^{-1}$. Above synchronization transition, only regular dynamics have been observed. However, for $N \gg 1$ and close to the synchronization transitions, the regime is complex: if it is not chaotic, then it is quasiperiodic with a large number of incommensurate frequencies; here, statistical approaches based on finite-size scaling have been applied to find the scaling form of N -dependence of the order parameters.^{47,48}

- (g) *Kuramoto model on networks.* Kuramoto model has been initially formulated for the ensemble of globally coupled oscillators. Recently, it has been extensively studied for other coupling configurations, i.e., for networks of different complexity, including small-world networks,⁴⁹ multidimensional hypercubic lattices,⁴⁸ networks with a modular structure,⁵⁰ and an ensemble with an extra leading element (hub).⁵¹ Dynamical properties of the transition to synchronization depend on the network topology.⁵² One of the popular applications here is study of synchronization of power grids.⁵³
- (h) *External forcing and collective phase resetting.* As one can see from Eq. (20), the macroscopic order parameter obeys an equation for a self-sustained oscillator. Thus, the collective mode has the same properties as such an oscillator. In particular, if the ensemble is forced by a periodic force, the latter can entrain the frequency of the mean field oscillations.^{28,54} If the force is represented by a pulse train, then each pulse shifts the phases of all oscillators and, hence, the phase of the collective mode as well; then, in analogy to the case of an isolated oscillator, one defines the phase response curve for the collective mode as a dependence of the phase shift of the mode on the phase when the pulse stimulation occurred.⁵⁵
- (i) *Mathematical results.* The Kuramoto model has been extensively studied on the physical and computational level, but rigorous mathematical results for the thermodynamic limit are sparse. They mainly refer to the stability analysis of the desynchronized state,^{30,56} for a description of the synchronization transition as a bifurcation problem see Ref. 57.

IV. GENERAL COUPLING FUNCTIONS

Now we come back to Eq. (2). For a general coupling function g , this equation represents the Daido model.^{58,59} Expanding g into Fourier series, $g(\eta) = \sum_n g_n e^{in\eta}$, and introducing generalized order parameters

$$Z_n = N^{-1} \sum_{j=1}^N e^{in\varphi_j}, \quad (21)$$

one can re-write the model as

$$\dot{\varphi}_k = \omega_k + \varepsilon \sum_n g_n Z_n e^{-in\varphi_k}. \quad (22)$$

One can see that generally all order parameters should be determined self-consistently, so that a complete analysis at general coupling is still missing. We mention here several interesting regimes appearing due to high harmonics in coupling function.

- (j) *Clustering.* Even for identical oscillators, the WS theory does not apply, and a population can build several clusters,⁶⁰ each of them consisting of fully synchronized units.

- (k) *Heteroclinic cycles.* In Ref. 61, nontrivial regimes of clustering and switching between different cluster states have been found in a population of identical oscillators with a function g containing the first and the second harmonics. This complex dynamics is due to a heteroclinic cycle in the phase space, well understood for small networks.⁶²
- (l) *Multi-branch entrainment.* Already for two harmonic components in the coupling function g , the r.h.s. of Eq. (22) as a function of φ can have two stable branches. This is a new situation compared to the standard Kuramoto model: now for a given mean field entrainment at two different microscopic phases is possible. This leads to an enormous multiplicity of microscopic states and to a complex structure of macroscopic regimes.^{59,63,64}

V. NON-PHASE MODELS

We started our discussion of collective ensemble dynamics from phase models. This approach relies on the well-known idea that motion along the limit cycle of an autonomous system can be parameterized by a single variable, the phase. Moreover, if the interaction of the oscillator with the environment is weak so that one can neglect the deviation of the trajectory from the cycle of the autonomous system, then the low-dimensional phase description remains valid. Generally, for strong coupling, one has to analyze full dynamical models which is a complicated problem that can hardly be treated analytically.

Numerical studies reveal that transition to collective synchrony is a quite general property, observed for various periodic, noisy, and even chaotic oscillators,^{5,65} including, e.g., spiking and bursting model neurons. The picture is, however, non-universal. The most transparent and studied model is an ensemble of coupled Stuart-Landau oscillators⁶⁶ (this oscillator is the simplest prototype of a limit-cycle oscillator, with a perfect separation of amplitude and phase variables, see Eq. (24) below), and essentially new effects are the oscillation quenching, when too strong coupling effectively introduces additional damping to ensemble elements, and collective chaos in a system of units that exhibit periodic oscillation in the absence of coupling.

On the other hand, non-identical chaotic phase-coherent Rössler oscillators adjust their frequencies and phases (this effect is known as phase synchronization of chaos⁶⁷) and produce a nearly periodic mean field. The oscillators themselves remain chaotic, but irregular fluctuations of the amplitudes turn to be averaged out in the mean field; the small fluctuations of the latter are presumably due to the finite-size effect.⁶⁵ Experimental studies on chaotic electrochemical oscillators⁵ confirm theoretical predictions.

Another important class is rotators: these systems are described by an angle-like variable, which is very similar to the phase, and do not have amplitudes. (Recall that true phase grows uniformly in time, while the time derivative of the angle variable depends on the variable itself.) If an equation governing the rotator's angle is one-dimensional, the dynamics can be reduced to a Kuramoto-type phase model,

what has been extensively discussed in the context of Josephson junctions.³⁸ Quite often the inertia of rotators cannot be neglected, and hence, they are described by a second-order equation for the angle variable. In this case, the dynamics can strongly deviate from that of the Kuramoto model, e.g., transition to synchrony may exhibit hysteresis, similarly to a first-order phase transition.⁶⁸ A particular subclass is constituted by globally coupled rotators without damping: this conservative system yields the so-called Hamiltonian Mean Field (HMF) model;⁶⁹ see Ref. 70 for a review of recent results and relation to the Kuramoto model.

VI. COMPLEX COLLECTIVE DYNAMICS AROUND SYNCHRONY

We have discussed in detail the main effect observed in globally coupled systems, namely, emergence of the collective mode, which is well-understood in the simplest case, when many units synchronize and therefore their outputs sum up coherently, contributing to this mode. We have also mentioned that the mode itself can exhibit chaotic dynamics. Now we discuss other, less explored, situations, when the collective dynamics of an ensemble is complex.

A. Partial synchrony

In case of the Kuramoto model of identical oscillators, only two situations are possible: full synchrony for attractive coupling (order parameter equal to one) and fully asynchronous state (zero order parameter) for repulsive coupling. However, this situation is not general and we can face a case, when both fully synchronous and fully asynchronous states are unstable, so that the system settles somewhere in between, at a state which is often called partial synchrony. The simplest and well-known example of partial synchrony is clustering; below we discuss several other situations where oscillators are not organized in clusters.

We exemplify partial synchrony with $N \gg 1$ mean field coupled oscillators.⁷¹

$$\begin{aligned}\dot{x}_k &= y_k - x_k^3 + 3x_k^2 - z_k + 5 + \varepsilon(X - x_k), \\ \dot{y}_k &= 1 - 5x_k^2 - y_k, \\ \dot{z}_k &= 0.006[4(x_k + 1.56) - z_k],\end{aligned}\quad (23)$$

where $k = 1, \dots, N$ and $X = N^{-1} \sum_{j=1}^N x_j$. Here, individual units represent a popular Hindmarsh-Rose neuronal model.⁷² The quite standard parameter values used here correspond to a limit-cycle solution for the uncoupled neurons, or, in neuroscience language, to a state of periodic spiking. Fully synchronous state, $x_1 = x_2 = \dots = x_N$, $y_1 = y_2 = \dots = y_N$, $z_1 = z_2 = \dots = z_N$ is obviously a solution of the system. However, stability of this state depends on the coupling strength ε , as can be checked numerically by virtue of computation of evaporation multipliers μ , related to the transversal Lyapunov exponents λ as $|\mu| = e^{\lambda T}$, where T is the oscillation period.^{73,74}

This analysis, confirmed by direct simulation, demonstrates that there exists a critical coupling value ε_c , at which the synchronous solution loses its stability via a Hopf-like

bifurcation, i.e., two complex multipliers cross the unit circle, giving birth to a new frequency. (Notice that the stability of the synchronous solution is re-gained for very large ε .) Beyond the synchrony breaking, the states of oscillators in the phase space form a thin stripe, stretched along the limit cycle; the points within this stripe slowly interchange their position, with a characteristic time of tens of periods. Roughly speaking, the average frequency of all oscillators and of the mean field is the same, but the phase shift of the oscillators with respect to the field is slowly modulated. As a result, the dynamics looks quite complicated and is possibly weakly chaotic.⁷¹

For another example, we consider a popular model of a series array of resistively shunted Josephson junctions. The junctions are coupled by virtue of a parallel RLC -load.³⁸ As has been shown in Ref. 38, for a weak coupling and linear load, the system is equivalent to the Kuramoto model. Consider now a *nonlinear coupling*; namely, let the inductance in the RLC -circuit be nonlinear so that the magnetic flux depends on the current \dot{Q} through the RLC -load as $\Phi = L_0 \dot{Q} + L_1 \dot{Q}^3$. Numerical study⁷⁴ reveals a synchrony-breaking transition: at $\varepsilon = \varepsilon_c$ the synchronous state becomes unstable; at this critical coupling value real evaporation multiplier μ becomes larger than one (in contradistinction to example (23) where the multipliers are complex). For $\varepsilon > \varepsilon_c$, the systems are in a state of partial synchrony, with the order parameter being a smooth decreasing function of ε . Furthermore, the dynamics becomes quasiperiodic: the frequency of the mean field is larger than the frequency of individual junctions and these frequencies are generally incommensurate, so that the junctions are not locked to the field. The frequency difference grows with $\varepsilon - \varepsilon_c$.

A general theory of partially synchronous states which appear after the synchrony breaking is still missing and requires further investigations. Below we present an analytically tractable and relatively transparent example.

B. Self-organized quasiperiodic dynamics

Now we analyze the system of $N \gg 1$ Stuart-Landau oscillators

$$\begin{aligned}\dot{a}_k &= [1 + i(\omega + \kappa)]a_k - (1 + i\kappa)|a_k|^2 a_k \\ &+ (\xi_1 + i\xi_2)A - (\eta_1 + i\eta_2)|A|^2 A,\end{aligned}\quad (24)$$

where $A = N^{-1} \sum_{j=1}^N a_j$ and $\xi_{1,2}, \eta_{1,2}$ are coupling parameters. This model differs from that of Ref. 66 due to the nonlinearity in coupling. In the weak coupling approximation, the model with purely linear coupling⁶⁶ (i.e., with $\eta_1 = \eta_2 = 0$) yields the Kuramoto-Sakaguchi Eq. (4), while the nonlinear Eq. (24) reduces to a particular case of the following phase model:⁷⁵

$$\dot{\varphi} = \omega + \varepsilon \alpha(\varepsilon, R) \sin[\Theta - \varphi_k + \beta(\varepsilon, R)].\quad (25)$$

Equation (25) can be considered as an extension of the Kuramoto-Sakaguchi model. Here, R is the mean field amplitude and the bifurcation parameter ε corresponds to a combination of the parameters $\xi_{1,2}, \eta_{1,2}$ and $\alpha(\varepsilon, R), \beta(\varepsilon, R)$ are some functions. Notice that Eq. (25) appears also in a model

of Stuart-Landau oscillators coupled via a common nonlinear medium,^{74,75} similarly to the Josephson junction model.

Let us consider the effect of these functions separately, starting with the case when $\beta = \text{const}$, $|\beta| < \pi/2$, i.e., the coupling is attractive, cf. Refs. 76 and 21. Suppose that α is a decreasing function of ε , e.g., $\alpha(\varepsilon, R) = (1 - \varepsilon R^2)R$. (This function indeed appears for a certain combination of parameters in Eq. (24).) For $\varepsilon < \varepsilon_c = 1$, this function is positive for fully synchronous case $R = 1$ and, hence, this state is stable. For $\varepsilon > \varepsilon_c$, we have $\alpha(\varepsilon, 1) < 0$, i.e., the coupling becomes repulsive. As a result, the system stays at the border between attraction and repulsion, determined by the condition $\varepsilon R^2 = 1$, forming a *self-organized bunch state*.⁷⁵ In this state, for general initial conditions, the oscillator phases spread around the unit circle so that $R = 1/\sqrt{\varepsilon}$; this bunch is stationary in the coordinate frame, rotating with the frequency ω .

Now we analyze a more interesting case when $\alpha = R$ and $\beta(\varepsilon, R) = \beta_0 + \beta_1 \varepsilon^2 R^2$, $|\beta_0| < \pi/2$, $\beta_1 > 0$. It is easy to see that synchrony ($R = 1$) is stable if $\beta_0 + \beta_1 \varepsilon^2 < \pi/2$ and becomes unstable when ε exceeds the critical value $\varepsilon_c = \sqrt{(\pi/2 - \beta_0)/\beta_1}$. Again, the system settles at the border of stability, so that the condition $\beta_0 + \beta_1 \varepsilon^2 R^2 = \pi/2$ is fulfilled. However, in this case, the dynamics exhibits a peculiar feature, also possessed by the Josephson junction model discussed above: the frequency of the mean field differs from the frequency of the individual units. Thus, the state can be characterized by two generally irrationally related frequencies, and therefore, is denoted as self-organized quasiperiodicity (SOQ). Qualitatively, the emergence of quasiperiodic motion can be explained as follows. For $\varepsilon > \varepsilon_c$, the system is partially synchronous, i.e., $R < 1$ and all oscillators have different phases. (Notice that for general initial conditions without clusters the phases must be different according to Eq. (7).) Hence, the instantaneous frequencies of oscillators differ, and therefore, a stationary (in a rotating coordinate frame) distribution is not a solution. Quantitative analysis of SOQ dynamics with computation of the frequencies of the collective mode and of the oscillators can be found in Refs. 74 and 75. SOQ states in real-world systems have been demonstrated in experiments with electronic circuits with global nonlinear coupling.⁷

To complete the discussion of this issue, we mention that similar complex states with a nonzero collective mode can appear also without desynchronization transition. For some systems, the fully synchronous and the completely asynchronous states are unstable already for infinitely small coupling. A well-studied example is the van Vreeswijk model of globally coupled leaky integrate-and-fire neurons.⁷⁷ Another example is the emergence of dephasing and bursting in a system of Morris-Lecar neuronal models;⁷⁸ computation of the evaporation multipliers for this system shows that the synchronous state is unstable for the positive coupling range, where the effect is observed.

C. Chimera-like states in globally coupled systems

We have already mentioned a symmetry-breaking chimera state in a system of two interacting Kuramoto

populations. Now we discuss emergence of a chimera-like state in a single homogeneous population. At first glance, such regimes do not seem to be allowed, because identical units subject to a common force should exhibit the same dynamics (i.e., to be either all synchronized or all desynchronized). On the other hand, there is a number of numerical observations of the mixed states, where only a fraction of the population merges to one or several clusters, while other elements remain scattered,⁷⁹ see also recent studies of chimera states in linearly and nonlinearly coupled Stuart-Landau oscillators.⁸⁰ The conditions for emergence of such mixed states are not yet fully clear. Nevertheless, we can outline one mechanism which results in splitting of the population into coherent and incoherent groups.

Identical elements subject to the same force can behave differently if they are bistable (multistable), i.e., possess at least two nontrivial attractors. Another requirement is that for the units on one attractor the mean field coupling shall be attracting so that this group synchronizes, while for the elements on the other attractor the coupling is repulsive, so that these elements form an incoherent group. Taking into account that this partially coherent and partially incoherent state shall be maintained self-consistently, we conclude that this condition is not trivial. To illustrate this mechanism, we first consider a rather artificial but transparent example,⁸¹ where the oscillators are the modified Stuart-Landau systems with the amplitude-dependent oscillation frequency. The modification refers to the nonlinearity: in addition to the 3rd order term $|a_k|^2 a_k$, we add also the terms $\sim |a_k|^4 a_k$, $\sim |a_k|^6 a_k$, cf. Eq. (24). As a result, the systems possess two stable limit cycles with different frequencies, $\Omega_2 > \Omega_1$. The global coupling has its own dynamics, cf. Refs. 38 and 82, so that the mean field forces a harmonic oscillator, $\ddot{u} + \gamma \dot{u} + \eta^2 u = N^{-1} \sum_j^N \text{Re}(a_j)$, and its output \dot{u} acts on the Stuart-Landau systems. Suppose now that parameters are chosen in such a way that $\Omega_1 < \eta < \Omega_2$. Since the phase shift introduced by the harmonic oscillator in the last equation is frequency-dependent, the coupling synchronizes the large amplitude limit-cycle oscillations but prevents synchronization of the low-amplitude ones.

Much less trivial is emergence of coherent-incoherent states in an ensemble of globally coupled oscillators with internal delayed feedback loop. Such oscillators naturally appear, e.g., in laser optics⁸³ as well as in numerous biological applications.⁸⁴ In the simplest case, the autonomous dynamics of an oscillator with a delayed feedback loop can be described by a phase model, $\dot{\varphi} = \omega + \alpha \sin(\varphi_\tau - \varphi)$, where $\varphi_\tau \equiv \varphi(t - \tau)$, τ is the delay, and α quantifies the feedback strength.^{83,85} Assuming the global coupling in the ensemble of such oscillators to be of the Kuramoto-Sakaguchi type, one writes the model as⁸¹

$$\dot{\varphi}_k = \omega + \alpha \sin(\varphi_{\tau,k} - \varphi_k) + \varepsilon R \sin(\Theta - \varphi_k + \beta). \quad (26)$$

Stability analysis of the fully synchronous one-cluster state yields that it is unstable for $\beta > \pi/2$. However, numerical simulation shows that for $\beta \geq \pi/2$ the asynchronous state with zero mean field, $R = 0$, is also unstable (see Ref. 81 for other parameter values). Thus, the system “chooses” a state

between full coherence and full incoherence, and in a range of parameters, this state is reminiscent of a chimera: there is one big cluster (about 70% of the population size) and a cloud of asynchronous oscillators. The frequency of the cluster is larger than the frequency of the cloud oscillators. Noteworthy, these states appear in the case when autonomous oscillators are monostable, though close to the bistability domain (for sufficiently strong feedback, $\alpha\tau > 1$, autonomous time-delayed unit admits multiple periodic solutions). It means that the systems become bistable due to coupling and, thus, the chimera-like state emerges due to the dynamically sustained bistability.

VII. AN IMPORTANT APPLICATION: NEUROSCIENCE

Within fifty years since Wiener hypothesized the role of the collective synchrony in the brain dynamics, importance of this phenomenon for neuroscience became highly recognized. It is now well-accepted that macroscopic neural rhythms appear due to coordination of firing and/or bursting of interconnected neuronal network and the Kuramoto model, in spite of its simplicity, became a paradigmatic and widely used setup in this field, see Ref. 86 for a review. Noteworthy, the neuroscience applications turned out to be very fruitful for the theoretical development, posing quite interesting, from the viewpoint of nonlinear dynamics, problems. In particular, in the context of complex dynamical states which are of general interest, we have several times mentioned neuronal models, e.g., the Hindmarsh-Rose one. Below, without an ambition to provide a comprehensive review, we outline several relevant issues.

For neuronal models, synchronization in a fully connected network was observed for map-based and continuous time models, both for the regimes of periodic spiking and bursting, see, e.g., Ref. 87. A bit more detailed models consist of excitatory and inhibitory neuronal subpopulations.⁸⁸ Another issue is a detailed description of synaptic coupling,⁸⁹ also with account of plasticity.⁹⁰

Fully connected network is certainly a rather crude approximation for a neuronal population. However, in some cases it is quite reasonable, as indicated by a numerical study of randomly coupled networks of map-based neurons:⁸⁸ if every unit is connected only to 0.5% of elements in the ensemble, then synchronization properties are practically indistinguishable from the fully connected case. For many problems, the assumption of random connectivity is not appropriate; in this case, one has to use physiologically motivated connectivity structure. An interesting approach has been recently suggested to treat large random networks of coupled oscillators.⁹¹ To adequately perform the thermodynamic limit and preserve disorder due to randomness of connections, a heterogeneous mean-field approach has been developed in which disorder remains the same while the size of the system grows. This approach yields a description of both microscopic and global features of neuronal synchrony in the model.⁹¹ Another interesting observation relates to diluted random networks of spike-coupled neurons:⁹² while for weak coupling synchrony establishes quite quickly, for large coupling a very long (in fact, exponential in the

network size) transient disordered state is observed, characterized by a negative largest Lyapunov exponent (so-called “stable chaos”).

A. Control of collective synchrony

As an example of a particular application of the discussed theoretical ideas, we address the problem of suppression of the collective synchrony. This problem is relevant for a medical technique, called deep brain stimulation (DBS). This technique is exploited to treat the Parkinson’s disease if it cannot be cured by medication. The DBS implies electrical stimulation of deep brain structures through the implanted micro-electrodes. The modern devices use the constant frequency (ca. 100 – 120 Hz) pulse stimulation, which is typically applied around the clock. Although the exact mechanisms of DBS are not yet quite understood, it is widely used to reduce the limb tremor.

Analysis of electrical or magnetic brain activity shows that Parkinsonian tremor is associated with the pronounced spectral power at 10–12 Hz.⁹³ Hence, it is reasonable to hypothesize that this pathological rhythm emerges due to synchronization in a neuronal population and to consider the DBS as a desynchronization problem.⁹⁴ The main idea is then to develop efficient techniques that reliably suppress the unwanted activity with minimal stimulation. There are several directions in these studies. The first one implies open-loop stimulation with specially organized pulses⁹⁵ through several closely spaced sites; the main assumption is that these pulses cause formation of several clusters shifted in phase with respect to each other, so that summation of their oscillations results in a vanishing mean field.

Another direction in control of collective oscillation, based on the closed-loop feedback, was suggested in Refs. 96 and 87 and verified in experiments with electrochemical oscillators in Ref. 6. The idea is quite transparent. Consider a globally coupled system: all elements are forced by the collective mode and synchronize due to this forcing. Suppose we measure the collective oscillation and feed it back, shifting its phase and properly amplifying it so that the feedback signal exactly compensates the mean field. In this case, the oscillators become unforced and naturally desynchronize due to frequencies mismatch, internal noise, etc. Since the units desynchronize, the mean field tends to a constant, and so does the feedback signal. The constant component of the feedback signal can be easily eliminated; in this way, one performs a *vanishing stimulation* control. It means that stimulation tends to zero as soon as the undesired oscillation is suppressed; this property is extremely important for medical applications. The vanishing stimulation control can be implemented via a delayed feedback^{87,88} (see Ref. 97 for a variant with nonlinear delayed feedback which is however not vanishing) or via a phase shifting passive system without delay.⁹⁸ Moreover, an adaptive strategy can be implemented,⁹⁹ so that the desired state can be achieved in spite of unknown parameters of the system to be controlled. Notice that adjusting the phase shift introduced by the feedback loop one can, depending on the goal, suppress or enhance the collective synchrony.

VIII. SURPRISES AND OUTLOOK

In the first years after development of ideas of synchronization in large ensembles, especially after Y. Kuramoto constructed his self-consistent theory, it looked like the very phenomenon of synchronization transition is rather simple and universal. In this paper, we tried to show how in fact nontrivial is even the simplest setup: features like partial integrability, existence of an exact low-dimensional manifold, nontrivial transition scenarios even for unimodal distributions of frequencies, chimera states, self-organized quasiperiodicity occur already in the simplest case of sine-coupled phase oscillators. For generalizations of this model, one observes a plethora of dynamical phenomena which is still far from being exhausted. Furthermore, addition of complexity to the basic model, e.g., by consideration of coupled oscillators on networks, results in further growth of the diversity of possible regimes.

It seems to us that in the nearest future we will experience spreading of the synchronization theory far beyond its initial scope of nonlinear dissipative dynamical systems, e.g., to cover quantum objects.¹⁰⁰ On the experimental level, advanced methods of oscillators' control and of data analysis will possibly reveal microscopic details of nontrivial synchronization patterns. On the other hand, growing interest of mathematicians to these problems indicates that the field may become a part of mathematical physics as well.

¹E. Kaempfer, *The History of Japan (With a Description of the Kingdom of Siam)* (Sloane, London, 1727) posthumous translation; or reprint by McLehose, Glasgow, 1906.

²See <http://1000fireflies.net> for description of interacting bike safety lights and their synchronization when up to 250 cyclists ride together, as part of Chicago Artists Month.

³N. Wiener, *Cybernetics or Control and Communication in the Animal and the Machine* (MIT Press, Cambridge, MA, 1965).

⁴S. H. Strogatz, "Norbert Wiener's brain waves," in *Frontiers in Mathematical Biology*, Lecture Notes in Biomathematics, Vol. 100, edited by S. A. Levin (Springer-Verlag, Berlin-Heidelberg, 1994), pp. 122–138.

⁵I. Kiss, Y. Zhai, and J. Hudson, "Emerging coherence in a population of chemical oscillators," *Science* **296**, 1676–1678 (2002).

⁶Y. Zhai, I. Z. Kiss, and J. L. Hudson, "Control of complex dynamics with time-delayed feedback in populations of chemical oscillators: Desynchronization and clustering," *Ind. Eng. Chem. Res.* **47**, 3502 (2008).

⁷A. A. Temirbayev, Z. Z. Zhanabaev, S. B. Tarasov, V. I. Ponomarenko, and M. Rosenblum, "Experiments on oscillator ensembles with global nonlinear coupling," *Phys. Rev. E* **85**, 015204(R) (2012); A. A. Temirbayev, Y. D. Nalibayev, Z. Z. Zhanabaev, V. I. Ponomarenko, and M. Rosenblum, "Autonomous and forced dynamics of oscillator ensembles with global nonlinear coupling: An experimental study," *Phys. Rev. E* **87**, 062917 (2013).

⁸E. A. Martens, S. Thutupalli, A. Fourriere, and O. Hallatschek, "Chimera states in mechanical oscillator networks," *Proc. Natl. Acad. Sci. U. S. A.* **110**, 10563–10567 (2013).

⁹S. Benz and C. Burroughs, "Coherent emission from two-dimensional Josephson junction arrays," *Appl. Phys. Lett.* **58**, 2162–2164 (1991).

¹⁰K. Hirose, S. Kittaka, Y. Oishi, F. Kannari, and T. Yanagisawa, "Phase locking in a nd:yyo4 waveguide laser array using talbot cavity," *Opt. Express* **21**, 24952–24961 (2013).

¹¹P. Richard, B. M. Bakker, B. Teusink, K. V. Dam, and H. V. Westerhoff, "Acetaldehyde mediates the synchronization of sustained glycolytic oscillations in population of yeast cells," *Eur. J. Biochem.* **235**, 238–241 (1996).

¹²A. Prindle, P. Samayoa, I. Razinkov, T. Danino, L. S. Tsimring, and J. Hasty, "A sensing array of radically coupled genetic "biopixels,"" *Nature* **481**, 39–44 (2012).

¹³P. Dallard, T. Fitzpatrick, A. Flint, A. Low, R. R. Smith, M. Willford, and M. Roche, "London millennium bridge: Pedestrian-induced lateral vibration," *J. Bridge Eng.* **6**, 412–417 (2001).

¹⁴Z. Neda, E. Ravasz, Y. Brechet, T. Vicsek, and A.-L. Barabási, "Tumultuous applause can transform itself into waves of synchronized clapping," *Nature* **403**, 849–850 (2000); "Physics of the rhythmic applause," *Phys. Rev. E* **61**, 6987–6992 (2000).

¹⁵S. M. Henson, J. L. Hayward, J. M. Cushing, and J. C. Galusha, "Socially induced synchronization of every-other-day egg laying in a seabird colony," *The Auk* **127**, 571–580 (2010).

¹⁶M. K. McClintock, "Menstrual synchrony and suppression," *Nature* **229**, 244–245 (1971); A. Weller and L. Weller, "Menstrual synchrony under optimal conditions: Bedouin families," *J. Comp. Psychol.* **111**, 143–151 (1997); M. A. Arden and L. Dye, "The assessment of menstrual synchrony: Comment on Weller and Weller (1997)," *J. Comp. Psychol.* **112**, 323–324 (1998).

¹⁷A. Ziolkiewicz, "Menstrual synchrony: Fact or artifact?," *Human Nat.* **17**, 419–432 (2006).

¹⁸A. T. Winfree, "Biological rhythms and the behavior of populations of coupled oscillators," *J. Theor. Biol.* **16**, 15–42 (1967); *The Geometry of Biological Time* (Springer, Berlin, 1980).

¹⁹C. C. Canavier, "Phase response curve," *Scholarpedia* **1**(12), 1332 (2006).

²⁰J. Ariaratnam and S. Strogatz, "Phase diagram for the winfree model of coupled nonlinear oscillators," *Phys. Rev. Lett.* **86**, 4278–4281 (2001); D. Quinn, R. Rand, and S. Strogatz, "Singular unlocking transition in the winfree model of coupled oscillators," *Phys. Rev. E* **75**, 036218 (2007); D. Pazó and E. Montbrío, "Low-dimensional dynamics of populations of pulse-coupled oscillators," *Phys. Rev. X* **4**, 011009 (2014).

²¹F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro, and S. Stramaglia, "Phase diagram of a generalized Winfree model," *Phys. Rev. E* **75**, 051104 (2007).

²²Y. Kuramoto, "Self-entrainment of a population of coupled nonlinear oscillators," in *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes Physics, Vol. 39, edited by H. Araki (Springer, New York, 1975), p. 420; *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).

²³D. Pazó, "Thermodynamic limit of the first-order phase transition in the Kuramoto model," *Phys. Rev. E* **72**, 046211 (2005).

²⁴H. Sakaguchi and Y. Kuramoto, "A soluble active rotator model showing phase transition via mutual entrainment," *Prog. Theor. Phys.* **76**, 576–581 (1986).

²⁵S. Watanabe and S. H. Strogatz, "Integrability of a globally coupled oscillator array," *Phys. Rev. Lett.* **70**, 2391–2394 (1993); "Constants of motion for superconducting Josephson arrays," *Physica D* **74**, 197–253 (1994).

²⁶A. Pikovsky and M. Rosenblum, "Partially integrable dynamics of hierarchical populations of coupled oscillators," *Phys. Rev. Lett.* **101**, 264103 (2008); "Dynamics of heterogeneous oscillator ensembles in terms of collective variables," *Physica D* **240**, 872–881 (2011).

²⁷S. A. Marvel, R. E. Mirollo, and S. H. Strogatz, "Phase oscillators with global sinusoidal coupling evolve by Mobius group action," *Chaos* **19**, 043104 (2009).

²⁸E. Ott and T. M. Antonsen, "Low dimensional behavior of large systems of globally coupled oscillators," *Chaos* **18**, 037113 (2008).

²⁹E. Ott and T. M. Antonsen, "Long time evolution of phase oscillator systems," *Chaos* **19**, 023117 (2009).

³⁰R. E. Mirollo, "The asymptotic behavior of the order parameter for the infinite-N Kuramoto model," *Chaos* **22**, 043118 (2012).

³¹D. M. Abrams, R. Mirollo, S. H. Strogatz, and D. A. Wiley, "Solvable model for chimera states of coupled oscillators," *Phys. Rev. Lett.* **101**, 084103 (2008).

³²Y. Kuramoto and D. Battogtokh, "Coexistence of coherence and incoherence in nonlocally coupled phase oscillators," *Nonlinear Phenom. Complex Syst.* **5**, 380–385 (2002); D. M. Abrams and S. H. Strogatz, "Chimera states for coupled oscillators," *Phys. Rev. Lett.* **93**, 174102 (2004); O. E. Omel'chenko, Y. L. Maistrenko, and P. A. Tass, "Chimera states: The natural link between coherence and incoherence," *Phys. Rev. Lett.* **100**, 044105 (2008).

³³P. So and E. Barreto, "Generating macroscopic chaos in a network of globally coupled phase oscillators," *Chaos* **21**, 033127 (2011).

- ³⁴M. Komarov and A. Pikovsky, "Dynamics of multifrequency oscillator communities," *Phys. Rev. Lett.* **110**, 134101 (2013).
- ³⁵M. Komarov and A. Pikovsky, "Effects of nonresonant interaction in ensembles of phase oscillators," *Phys. Rev. E* **84**, 016210 (2011).
- ³⁶O. E. Omel'chenko and M. Wolfrum, "Nonuniversal transitions to synchrony in the Sakaguchi-Kuramoto model," *Phys. Rev. Lett.* **109**, 164101 (2012).
- ³⁷B. Eckhardt, E. Ott, S. H. Strogatz, D. M. Abrams, and A. McRobie, "Modeling walker synchronization on the millennium bridge," *Phys. Rev. E* **75**, 021110 (2007).
- ³⁸K. Wiesenfeld and J. W. Swift, "Averaged equations for Josephson junction series arrays," *Phys. Rev. E* **51**, 1020–1025 (1995); K. Wiesenfeld, P. Colet, and S. H. Strogatz, "Synchronization transition in a disordered Josephson series array," *Phys. Rev. Lett.* **76**, 404–407 (1996).
- ³⁹J. Grollier, V. Cros, and A. Fert, "Synchronization of spin-transfer oscillators driven by stimulated microwave currents," *Phys. Rev. B* **73**, 060409(R) (2006); V. Tiberkevich, A. Slavin, E. Bankowski, and G. Gerhart, "Phase-locking and frustration in an array of nonlinear spin-torque nano-oscillators," *Appl. Phys. Lett.* **95**, 262505 (2009); A. Pikovsky, "Robust synchronization of spin-torque oscillators with an LCR load," *Phys. Rev. E* **88**, 032812 (2013).
- ⁴⁰D. Iatsenko, P. V. E. McClintock, and A. Stefanovska, "Glassy states and superrelaxation in populations of coupled phase oscillators," *Nat. Commun.* **5**, 4118 (2014).
- ⁴¹V. Vlasov, E. E. N. Macau, and A. Pikovsky, "Synchronization of oscillators in a Kuramoto-type model with generic coupling," *Chaos* **24**, 023120 (2014).
- ⁴²H. Hong and S. H. Strogatz, "Kuramoto model of coupled oscillators with positive and negative coupling parameters: An example of conformist and contrarian oscillators," *Phys. Rev. Lett.* **106**, 054102 (2011).
- ⁴³D. H. Zanette, "Synchronization and frustration in oscillator networks with attractive and repulsive interactions," *EPL* **72**, 190 (2005).
- ⁴⁴Y. Maistrenko, B. Penkovsky, and M. Rosenblum, "Solitary state at the edge of synchrony in ensembles with attractive and repulsive interaction," *Phys. Rev. E* **89**, 060901(R) (2014).
- ⁴⁵W. Braun, A. Pikovsky, M. A. Matias, and P. Colet, "Global dynamics of oscillator populations under common noise," *EPL* **99**, 20006 (2012).
- ⁴⁶O. V. Popovych, Y. L. Maistrenko, and P. A. Tass, "Phase chaos in coupled oscillators," *Phys. Rev. E* **71**, 065201 (2005).
- ⁴⁷H. Daido, "Intrinsic fluctuations and a phase transition in a class of large population of interacting oscillators," *J. Stat. Phys.* **60**, 753–800 (1990).
- ⁴⁸H. Hong, H. Chaté, H. Park, and L.-H. Tang, "Entrainment transition in populations of random frequency oscillators," *Phys. Rev. Lett.* **99**, 184101 (2007).
- ⁴⁹H. Hong, M. Y. Choi, and B. J. Kim, "Synchronization on small-world networks," *Phys. Rev. E* **65**, 026139 (2002).
- ⁵⁰P. S. Skardal and J. G. Restrepo, "Hierarchical synchrony of phase oscillators in modular networks," *Phys. Rev. E* **85**, 016208 (2012).
- ⁵¹Y. Kazanovich and R. Borisjuk, "Synchronization in oscillator systems with a central element and phase shifts," *Prog. Theor. Phys.* **110**, 1047–1057 (2003).
- ⁵²A. Arenas, A. Díaz-Guilera, and C. J. Pérez-Vicente, "Synchronization reveals topological scales in complex networks," *Phys. Rev. Lett.* **96**, 114102 (2006).
- ⁵³M. Rohden, A. Sorge, M. Timme, and D. Witthaut, "Self-organized synchronization in decentralized power grids," *Phys. Rev. Lett.* **109**, 064101 (2012); Y. Susuki and I. Mezić, "Nonlinear koopman modes and coherency identification of coupled swing dynamics," *IEEE Trans. Power Syst.* **26**, 1894 (2011); S. Lozano, L. Buzna, and A. Díaz-Guilera, "Role of network topology in the synchronization of power systems," *Eur. Phys. J. B* **85**, 472 (2012); F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proc. Natl. Acad. Sci. U. S. A.* **110**, 2005 (2013); A. E. Motter, S. A. Myers, M. Anghel, and T. Nishikawa, "Spontaneous synchrony in power-grid networks," *Nat. Phys.* **9**, 191 (2013); P. H. J. Nardelli, N. Rubido, C. Wang, M. S. Baptista, C. Pomalaza-Raez, P. Cardieri, and M. Latva-Aho, "Models for the modern power grid," *Eur. Phys. J.* **223**, 2423 (2014); A. Gajduk, M. Todorovski, and L. Kocarev, "Stability of power grids: an overview," *Eur. Phys. J.* **223**, 2387 (2014); P. Menck, J. Heitzig, J. Kurths, and H. Schellnhuber, "How dead ends undermine power grid stability," *Nat. Commun.* **5**, 3969 (2014).
- ⁵⁴H. Sakaguchi "Cooperative phenomena in coupled oscillator systems under external fields," *Prog. Theor. Phys.* **79**, 39–46 (1988); L. M. Childs and S. H. Strogatz, "Stability diagram for the forced Kuramoto model," *Chaos* **18**, 043128 (2008); Y. Baibolatov, M. Rosenblum, Z. Z. Zhanabaev, M. Kyzgarina, and A. Pikovsky, "Periodically forced ensemble of nonlinearly coupled oscillators: from partial to full synchrony," *Phys. Rev. E* **80**, 046211 (2009).
- ⁵⁵Y. Kawamura, H. Nakao, K. Arai, H. Kori, and Y. Kuramoto, "Collective phase sensitivity," *Phys. Rev. Lett.* **101**, 024101 (2008); T.-W. Ko and G. B. Ermentrout, "Phase-response curves of coupled oscillators," *Phys. Rev. E* **79**, 016211 (2009); Z. Levnajić and A. Pikovsky, "Phase resetting of collective rhythm in ensembles of oscillators," *Phys. Rev. E* **82**, 056202 (2010).
- ⁵⁶S. H. Strogatz and R. E. Mirollo, "Stability of incoherence in a population of coupled oscillators," *J. Stat. Phys.* **63**, 613–635 (1991); B. Fernandez, D. Gérard-Varet, and G. Giacomin, "Landau damping in the Kuramoto model," e-print: [arXiv:1410.6006](https://arxiv.org/abs/1410.6006) [math.AP].
- ⁵⁷H. Chiba and I. Nishikawa, "Center manifold reduction for large populations of globally coupled phase oscillators," *Chaos* **21**, 043103 (2011); H. Chiba, "A proof of the Kuramoto conjecture for a bifurcation structure of the infinite-dimensional Kuramoto model," *Ergodic Theor. Dyn. Syst.* **35**, 762–834 (2015); H. Dietert, "Stability and bifurcation for the Kuramoto model," e-print: [arXiv:1411.3752](https://arxiv.org/abs/1411.3752) [math.AP].
- ⁵⁸H. Daido, "A solvable model of coupled limit-cycle oscillators exhibiting perfect synchrony and novel frequency spectra," *Physica D* **69**, 394–403 (1993); "Critical conditions of macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators," *Prog. Theor. Phys.* **89**, 929–934 (1993); "Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: Bifurcation of the order function," *Physica D* **91**, 24–66 (1996).
- ⁵⁹H. Daido, "Multi-branch entrainment and multi-peaked order-functions in a phase model of limit-cycle oscillators with uniform all-to-all coupling," *J. Phys. A: Math. Gen.* **28**, L151–L157 (1995).
- ⁶⁰K. Okuda, "Variety and generality of clustering in globally coupled oscillators," *Physica D* **63**, 424–436 (1993).
- ⁶¹D. Hansel, G. Mato, and C. Meunier, "Clustering and slow switching in globally coupled phase oscillators," *Phys. Rev. E* **48**, 3470–3477 (1993).
- ⁶²P. Ashwin, G. Orosz, J. Wordsworth, and S. Townley, "Dynamics on networks of cluster states for globally coupled phase oscillators," *SIAM J. Appl. Dyn. Syst.* **6**, 728–758 (2007); Ö. Karabacak and P. Ashwin, "Heteroclinic ratchets in networks of coupled oscillators," *J. Nonlinear Sci.* **20**, 105–129 (2010).
- ⁶³H. Daido, "Multibranch entrainment and scaling in large populations of coupled oscillators," *Phys. Rev. Lett.* **77**, 1406–1409 (1996).
- ⁶⁴M. Komarov and A. Pikovsky, "Multiplicity of singular synchronous states in the Kuramoto model of coupled oscillators," *Phys. Rev. Lett.* **111**, 204101 (2013); "The Kuramoto model of coupled oscillators with a bi-harmonic coupling function," *Physica D* **289**, 18–31 (2014).
- ⁶⁵A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, "Synchronization in a population of globally coupled chaotic oscillators," *Europhys. Lett.* **34**, 165–170 (1996).
- ⁶⁶N. Nakagawa and Y. Kuramoto, "Collective chaos in a population of globally coupled oscillators," *Prog. Theor. Phys.* **89**, 313–323 (1993); "From collective oscillations to collective chaos in a globally coupled oscillator system," *Physica D* **75**, 74–80 (1994).
- ⁶⁷M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, "Phase synchronization of chaotic oscillators," *Phys. Rev. Lett.* **76**, 1804–1807 (1996).
- ⁶⁸H. Tanaka, A. Lichtenberg, and S. Oishi, "First order phase transition resulting from finite inertia in coupled oscillator systems," *Phys. Rev. Lett.* **78**, 2104–2107 (1997).
- ⁶⁹M. Antoni and S. Ruffo, "Clustering and relaxation in Hamiltonian long-range dynamics," *Phys. Rev. E* **52**, 2361–2374 (1995).
- ⁷⁰A. Campa, T. Dauxois, and S. Ruffo, "Statistical mechanics and dynamics of solvable models with long-range interactions," *Phys. Rep.* **480**, 57–159 (2009); S. Gupta, A. Campa, and S. Ruffo, "Kuramoto model of synchronization: equilibrium and nonequilibrium aspects," *J. Stat. Mech.* **2014**, R08001.
- ⁷¹S. Ehrlich, A. Pikovsky, and M. Rosenblum, "From complete to modulated synchrony in networks of identical hindmarsh-rose neurons," *Eur. Phys. J. Spec. Top.* **222**, 2407–2416 (2013).
- ⁷²J. L. Hindmarsh and R. M. Rose, "A model for neuronal bursting using three coupled first order differential equations," *Proc. R. Soc. London, Ser. B* **221**, 87 (1984).
- ⁷³K. Kaneko, "Information cascade with marginal stability in a network of chaotic elements," *Physica D* **77**, 456–472 (1994); A. Pikovsky, O. Popovych, and Y. Maistrenko, "Resolving clusters in chaotic ensembles

- of globally coupled identical oscillators," *Phys. Rev. Lett.* **87**, 044102 (2001).
- ⁷⁴M. Rosenblum and A. Pikovsky, "Self-organized quasiperiodicity in oscillator ensembles with global nonlinear coupling," *Phys. Rev. Lett.* **98**, 064101 (2007).
- ⁷⁵A. Pikovsky and M. Rosenblum, "Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators," *Physica D* **238**(1), 27–37 (2009).
- ⁷⁶G. Filatrella, N. F. Pedersen, and K. Wiesenfeld, "Generalized coupling in the Kuramoto model," *Phys. Rev. E* **75**, 017201 (2007).
- ⁷⁷C. van Vreeswijk, "Partial synchronization in populations of pulse-coupled oscillators," *Phys. Rev. E* **54**, 5522–5537 (1996); P. Mohanty and A. Politi, "A new approach to partial synchronization in globally coupled rotators," *J. Phys. A: Math. Gen.* **39**, L415–L421 (2006).
- ⁷⁸S. K. Han, C. Kurrer, and Y. Kuramoto, "Dephasing and bursting in coupled neural oscillators," *Phys. Rev. Lett.* **75**, 3190–3193 (1995).
- ⁷⁹K. Kaneko, "Clustering, coding, switching, hierarchical ordering, and control in a network of chaotic elements," *Physica D* **41**, 137–172 (1990); H. Daido and K. Nakanishi, "Diffusion-induced inhomogeneity in globally coupled oscillators: Swing-by mechanism," *Phys. Rev. Lett.* **96**, 054101 (2006).
- ⁸⁰G. C. Sethia and A. Sen, "Chimera states: the existence criteria revisited," *Phys. Rev. Lett.* **112**, 144101 (2014); L. Schmidt, K. Schönleber, K. Krischer, and V. García-Morales, "Coexistence of synchrony and incoherence in oscillatory media under nonlinear global coupling," *Chaos* **24**, 013102 (2014).
- ⁸¹A. Yeldesbay, A. Pikovsky, and M. Rosenblum, "Chimeralike states in an ensemble of globally coupled oscillators," *Phys. Rev. Lett.* **112**, 144103 (2014).
- ⁸²S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, "Theoretical mechanics: Crowd synchrony on the Millennium Bridge," *Nature* **438**, 43–44 (2005).
- ⁸³C. Masoller, "Noise-induced resonance in delayed feedback systems," *Phys. Rev. Lett.* **88**, 034102 (2002).
- ⁸⁴L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rhythms of Life* (Princeton University Press, Princeton, NJ, 1988); A. Takamatsu, T. Fujii, and I. Endo, "Time delay effect in a living coupled oscillator system with the plasmodium of *physarum polycephalum*," *Phys. Rev. Lett.* **85**, 2026–2029 (2000); J. Batzel and F. Kappel, "Time delay in physiological systems: analyzing and modeling its impact," *Math. Biosci.* **234**, 61–74 (2011).
- ⁸⁵E. Niebur, H. G. Schuster, and D. M. Kammen, "Collective frequencies and metastability in networks of limit-cycle oscillators with time delay," *Phys. Rev. Lett.* **67**, 2753–2756 (1991); M. K. S. Yeung and S. H. Strogatz, "Time delay in the Kuramoto model of coupled oscillators," *Phys. Rev. Lett.* **82**, 648–651 (1999); D. Goldobin, M. Rosenblum, and A. Pikovsky, "Controlling oscillator coherence by delayed feedback," *Phys. Rev. E* **67**, 061119 (2003).
- ⁸⁶M. Breakspear, S. Heitmann, and A. Daffertshofer, "Generative models of cortical oscillations: neurobiological implications of the Kuramoto model," *Front. Human Neurosci.* **4**, 190 (2010).
- ⁸⁷M. G. Rosenblum and A. S. Pikovsky, "Delayed feedback control of collective synchrony: An approach to suppression of pathological brain rhythms," *Phys. Rev. E* **70**, 041904 (2004).
- ⁸⁸M. G. Rosenblum, N. Tukhlina, A. S. Pikovsky, and L. Cimponeriu, "Delayed feedback suppression of collective rhythmic activity in a neuronal ensemble," *Int. J. Bifurcation Chaos* **16**, 1989–1999 (2006).
- ⁸⁹N. Rulkov, I. Timofeev, and M. Bazhenov, "Oscillations in large-scale cortical networks: Map-based model," *J. Comp. Neurosci.* **17**, 203–222 (2004).
- ⁹⁰K. Mikkelsen, A. Imparato, and A. Torcini, "Sisyphus effect in pulse-coupled excitatory neural networks with spike-timing-dependent plasticity," *Phys. Rev. E* **89**, 062701 (2014); O. Popovych, S. Yanchuk, and P. Tass, "Self-organized noise resistance of oscillatory neural networks with spike timing-dependent plasticity," *Sci. Rep.* **3**, 2926 (2013).
- ⁹¹R. Burioni, M. Casartelli, M. di Volo, R. Livi, and A. Vezzani, "Average synaptic activity and neural networks topology: a global inverse problem," *Sci. Rep.* **4**, 4336 (2014).
- ⁹²R. Zillmer, R. Livi, A. Politi, and A. Torcini, "Desynchronization in diluted neural networks," *Phys. Rev. E* **74**, 036203 (2006).
- ⁹³P. Tass, M. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkman, A. Schnitzler, and H.-J. Freund, "Detection of $n:m$ phase locking from noisy data: Application to magnetoencephalography," *Phys. Rev. Lett.* **81**, 3291–3294 (1998).
- ⁹⁴P. A. Tass, *Phase Resetting in Medicine and Biology. Stochastic Modelling and Data Analysis* (Springer-Verlag, Berlin, 1999); "A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations," *Biol. Cybern.* **89**, 81–88 (2003); B. Lysyansky, O. V. Popovych, and P. A. Tass, "Desynchronizing anti-resonance effect of $m:n$ on-off coordinated reset stimulation," *J. Neural Eng.* **8**, 036019 (2011).
- ⁹⁵B. Lysyansky, O. V. Popovych, and P. A. Tass, "Multi-frequency activation of neuronal networks by coordinated reset stimulation," *Interface Focus* **1**, 75–85 (2010).
- ⁹⁶M. G. Rosenblum and A. S. Pikovsky, "Controlling synchrony in ensemble of globally coupled oscillators," *Phys. Rev. Lett.* **92**, 114102 (2004).
- ⁹⁷O. V. Popovych, C. Hauptmann, and P. A. Tass, "Effective desynchronization by nonlinear delayed feedback," *Phys. Rev. Lett.* **94**, 164102 (2005).
- ⁹⁸N. Tukhlina, M. Rosenblum, A. Pikovsky, and J. Kurths, "Feedback suppression of neural synchrony by vanishing stimulation," *Phys. Rev. E* **75**, 011918 (2007).
- ⁹⁹G. Montaseri, M. Javad Yazdanpanah, A. Pikovsky, and M. Rosenblum, "Synchrony suppression in ensembles of coupled oscillators via adaptive vanishing feedback," *Chaos* **23**, 033122 (2013).
- ¹⁰⁰P. Walker, T. C. H. Liew, D. Sarkar, M. Durska, A. P. D. Love, M. S. Skolnick, J. S. Roberts, I. A. Shelykh, A. V. Kavokin, and D. N. Krizhanovskii, "Suppression of Zeeman splitting of the energy levels of exciton-polariton condensates in semiconductor microcavities in an external magnetic field," *Phys. Rev. Lett.* **106**, 257401 (2011); M. Xu, D. Tieri, E. Fine, J. K. Thompson, and M. Holland, "Synchronization of two ensembles of atoms," *Phys. Rev. Lett.* **113**, 154101 (2014).