DAMOCO Toolbox
Brief illustration to the theory

Protophases and phases

Michael Rosenblum

Bjoern Kralemann **Arkady Pikovsky**

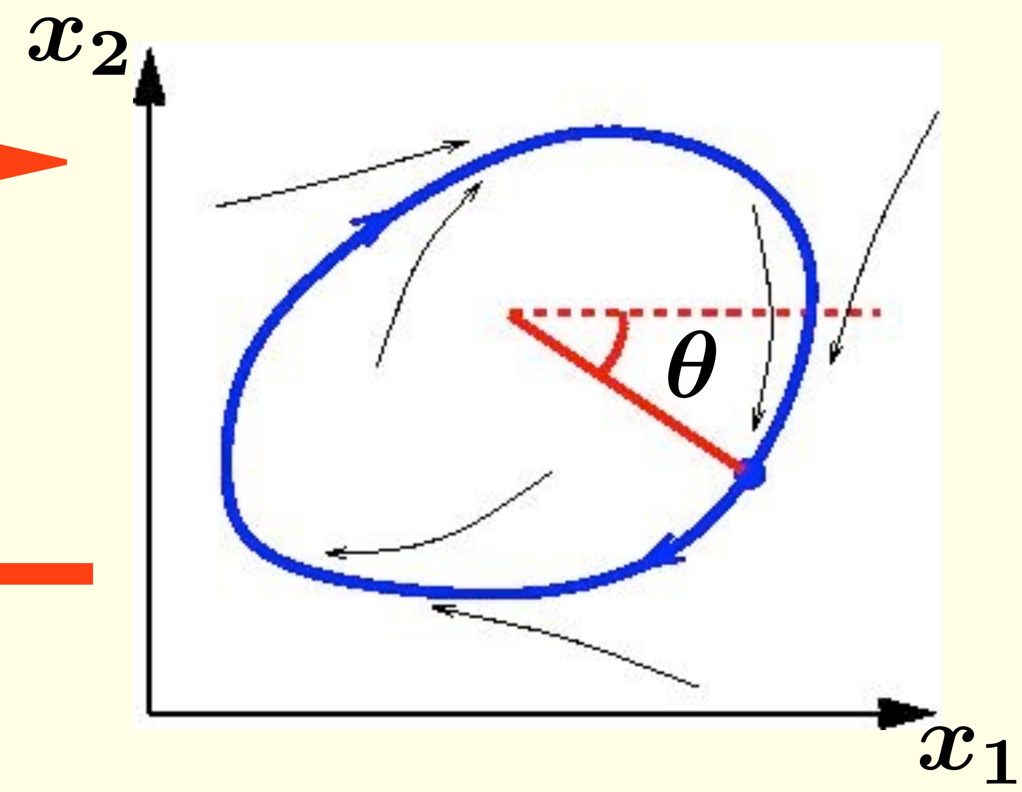
Institute of Physics and Astronomy
Potsdam University, Germany

Autonomous oscillator: Protophase and Phase

Limit cycle in the phase space

Equations $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$

Protophase θ



Protophase is a variable along the limit cycle, such that

- it grows monotonically, but generally not uniformly, i.e. $\dot{\theta} = f(\theta)$
- gains 2π with each rotation
- depends on the choice of observables, embedding, etc, i.e. is not invariant

From protophase to phase

Phase is a variable along the limit cycle, such that

- it ***grows uniformly***, $\dot{\varphi} = \omega = \text{const}$
- gains $\boldsymbol{2\pi}$ with each rotation
- does not depend on the choice of observables, embedding, etc, i.e. is invariant

Phase can be obtained from a protophase by a simple transformation $\boldsymbol{\theta} \rightarrow \boldsymbol{\phi(\theta)}$:

$$\frac{d\phi}{dt} = \omega \quad \Rightarrow \quad \frac{d\phi}{d\theta}\frac{d\theta}{dt} = \omega \quad \Rightarrow \quad \frac{d\phi}{d\theta} = \frac{\omega}{\dot{\theta}}$$

or, in integral form: $$\phi = \omega \int_0^\theta \frac{d\theta}{\dot{\theta}}$$

From time series to phase in two steps

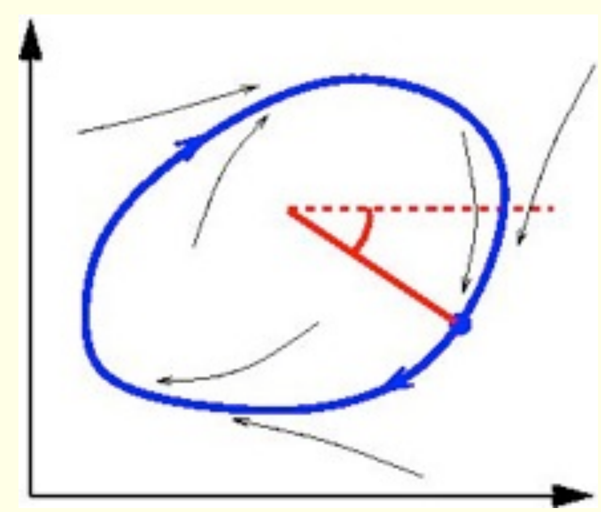
1. From an oscillatory signal a protophase can be obtained, e.g., via the Hilbert transform, complex wavelet transform, etc
2. Phase is obtained from the protophase by means of the transformation $\theta \rightarrow \phi(\theta)$

How to perform this transformation for noisy data?

For noise-free system we have: $$\frac{d\phi}{d\theta} = \frac{\omega}{\dot{\theta}}$$

We average the r.h.s.: $$\frac{d\phi}{d\theta} = \omega \left\langle \frac{dt}{d\theta}(\theta) \right\rangle_\theta = \sigma(\theta)$$

$(2\pi)^{-1}\sigma(\theta)$ is the probability density function of θ



Computing transformation function $\sigma(\theta)$

Estimation of the probability density is a standard problem of data analysis, e.g., it can be written as an integral along the trajectory (Kralemann et al., Phys. Rev. E, 2008):

$$(2\pi)^{-1}\sigma(\theta) = \frac{1}{T}\int_0^T \delta(\Theta(t) - \theta)\,dt$$

Writing $\sigma(\theta)$ as a Fourier series:

$$\sigma(\theta) = \sum_n S_n e^{in\theta}, \qquad S_n = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\theta) e^{-in\theta} d\theta$$

we get $$S_n = T^{-1}\int_0^T e^{-in\Theta(t)} dt$$

Transformation function $\sigma(\theta)$

With $\sigma(\theta) = \sum_n S_n e^{in\theta}$ the final transformation reads

$$\phi = \int_0^\theta \sigma(\theta')d\theta' = \theta + 2\sum_{n=1}^{\infty} \mathrm{Im}\left[\frac{S_n}{n}(e^{in\theta} - 1)\right]$$

For a time series of N points Θ_k,

the Fourier coefficients of $\sigma(\theta)$ are:

$$S_n = N^{-1}\sum_{k=1}^{N} e^{-in\Theta_k}$$

Optimal number of Fourier modes: C. Tenreiro, J. Nonparam. Statistics, 2011