DAMOCO Toolbox

Brief illustration to the theory

Phase dynamics of coupled oscillators

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Phase dynamics of two coupled oscillators (see, e.g., Kuramoto, 1984)

autonomous frequencies

coupling functions

phases

Phase equations
$$\boxed{\frac{d\varphi_{1,2}}{dt} = \omega_{1,2} + q_{1,2}(\varphi_{1,2}, \varphi_{2,1})}$$

can be derived from the full equations by means of a perturbation approach, i.e. in the weak coupling approximation

In typical cases $$q_1(\varphi_1, \varphi_2) = Z_1(\varphi_1) I_2(\varphi_2)$$

phase response curve (PRC)

forcing

Network of coupled oscillators

- Individual oscillator: $\dot{\mathbf{x}}_k = \mathbf{G_k}(\mathbf{x_k})$
  - limit cycle, parameterized by phase φ_k
  - phase grows linearly with time: $\dot{\varphi}_k = \omega_k = \text{const}$
- A network of N coupled oscillators

$$\dot{\mathbf{x}}_k = \mathbf{G_k}(\mathbf{x_k}) + \varepsilon \mathbf{H}_k(\mathbf{x_1}, \mathbf{x_2}, \ldots)$$

- If $\mathbf{x}_l$ enters the equation for $\mathbf{x}_k$ then there is a direct structural connection $l \to k$
- If $\mathbf{H}_k = \sum_{j \neq k} \mathbf{H}_{kj}(\mathbf{x_k}, \mathbf{x_j})$ then coupling is pairwise
- If there are terms $\mathbf{H}_{kjl}(\mathbf{x_k}, \mathbf{x_j}, \mathbf{x_l})$: cross-coupling

Weak coupling: Phase description

- Weak coupling, no synchrony: motion on the ***N-torus*** in the phase space of the full system
- This motion can be parameterized by ***N*** phases:

$$\dot{\varphi}_k = \omega_k + q_k(\varphi_1, \varphi_2, \ldots), \quad k = 1, \ldots, N$$

- New coupling functions q_k can be obtained by a perturbative reduction (Kuramoto 84):

$$q_k(\varphi_1, \varphi_2, \ldots) = \varepsilon q_k^{(1)}(\varphi_1, \varphi_2, \ldots) + \varepsilon^2 q_k^{(2)}(\varphi_1, \varphi_2, \ldots) + \ldots$$

- Pairwise coupling in the full system:
  - first-order approximation: pairwise terms like $\varepsilon q_{kl}^{(1)}(\varphi_k, \varphi_l)$
  - high-order approximation: ***terms, depending on many phases***, not only on the phases of directly coupled nodes

Important remark I

Phase equations can be analytically derived in the
<u>weak coupling approximation</u>

However, they are valid as long as the motion is quasiperiodic.
Even if they cannot be derived, they can be obtained from data!

As long as the systems remain asynchronous and not chaotic,
the phase space trajectory lies on the torus $\longrightarrow$
the motion can be parameterized by two phases

Hence, validity of our approach goes beyond the weak
coupling approximation!

Important remark II

Phase equations are valid also for
transients, not only for steady state

Hence, our approach does not require stationarity!