Spreading of energy in the Ding-Dong model

S. Roy[1,2] and A. Pikovsky[1]
[1]*Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, 14476 Potsdam, Germany*
[2]*Department of Physics, Indian Institute of Technology, Kanpur, India*

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We study the properties of energy spreading in a lattice of elastically colliding harmonic oscillators (Ding-Dong model). We demonstrate that in the regular lattice the spreading from a localized initial state is mediated by compactons and chaotic breathers. In a disordered lattice, the compactons do not exist, and the spreading eventually stops, resulting in a finite configuration with a few chaotic spots. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3695369]

In a linear medium, energy propagates in the form of waves, e.g., as phonons in a chain of linearly interacting particles. Disorder leads to Anderson localization which blocks the spreading. In a nonlinear medium, chaos may appear, so the Anderson localization is destroyed and a weak subdiffusive spreading of energy is observed. Still, it is not clear how the final stage of this very slow process looks like. There is a class of systems where the interaction between particles is purely nonlinear, here phonons do not exist and all spreading mechanisms are essentially nonlinear. In such ordered strongly nonlinear lattices, nonlinear waves—compactons—may transport energy to large distances, while in disordered case, one observes a slow subdiffusive energy spreading due to chaos. The Ding-Dong model that we study in this paper is a simple although singular realization of a strongly nonlinear lattice, where linear oscillators interact due to elastic collisions. Numerical simulations of this system are very effective and allow us to characterize the properties of compactons and energy spreading in disordered lattices at very large times. Our conclusion is that in the disordered case the spreading eventually stops, resulting in a few chaotic spots where 3-4 neighboring particles collide in an irregular manner.

I. INTRODUCTION

Dynamics of nonlinear lattices is one of the central topics of nonlinear science. Since the pioneering works of Fermi, Pasta, and Ulam (see Refs. 1 and 2) enormous progress have been achieved in understanding of different aspects of interesting dynamical properties of such systems, in particular of solitary waves (solitons and compactons), thermalization, and transition to chaos. Very popular in these studies are models of elastically colliding particles. Indeed, such systems have been proved to be tractable in deriving foundations of statistical mechanics from dynamical equations.[3] In the context of one-dimensional lattices, in papers[4,5] a so-called "Ding-a-Ling" model has been introduced, consisting of an alternate sequence of equal mass, hard point, free and bounded by harmonic forces elastically colliding particles. For further studies of this model, see Refs. 6 and 7.

In this paper, we study a more simple Ding-Dong model, introduced by Prosen and Robnik.[8] This is a lattice of colliding harmonic oscillators. Our main interest is in the general features of energy spreading in such lattices: how an initially localized perturbation (wave packet) spreads. We will see that this process can be regarded as an interplay of compactons (strongly localized solitary waves) and chaos.

The problem of energy spreading in nonlinear lattices has attracted large interest recently in the context of disordered systems. It is known that linear disordered lattices demonstrate Anderson localization, i.e., the spreading is effectively blocked by disorder. Nonlinearity typically destroys the localization, leading to a very slow energy spreading.[9–21] One of the still unsolved issues here is whether the spreading occurs indefinitely (although slowly) or is eventually stopped. The last possibility might be reasonable, as in course of spreading the energy density decreases and the system becomes closer and closer to the linear one. In general nonlinear disordered lattices, the answer to this dilemma is very difficult and relies on a heavy numerical integration (cf. Refs. 20 and 21). In this paper, we attack this problem for the Ding-Dong model, where numerical simulations are much more effective.

The paper is organized as follows. First, we introduce the Ding-Dong model. Then, we discuss compactons and chaotic breathers. Spreading of energy is studied next for homogeneous and disordered lattices. We conclude with a discussion.

II. DING-DONG MODEL

The Ding-Dong model has been first formulated and studied by Prosen and Robnik.[8] It is a chain of harmonic oscillators with a Hamiltonian

$$H = \sum_k \left(\frac{p_k^2}{2} + \frac{q_k^2}{2} \right). \tag{1}$$

The oscillators are aligned along a line with a spacing distance 1, so that if $q_k = 1 + q_{k+1}$, an elastic collision between the oscillators k and $k+1$ occurs, at which they exchange their momenta $p_k \to p_{k+1}, p_{k+1} \to p_k$. Together with the total energy, the energy of the center of mass motion $E_{cm} = \frac{1}{2}((\sum_k p_k)^2 + (\sum_k q_k)^2)$ is conserved.

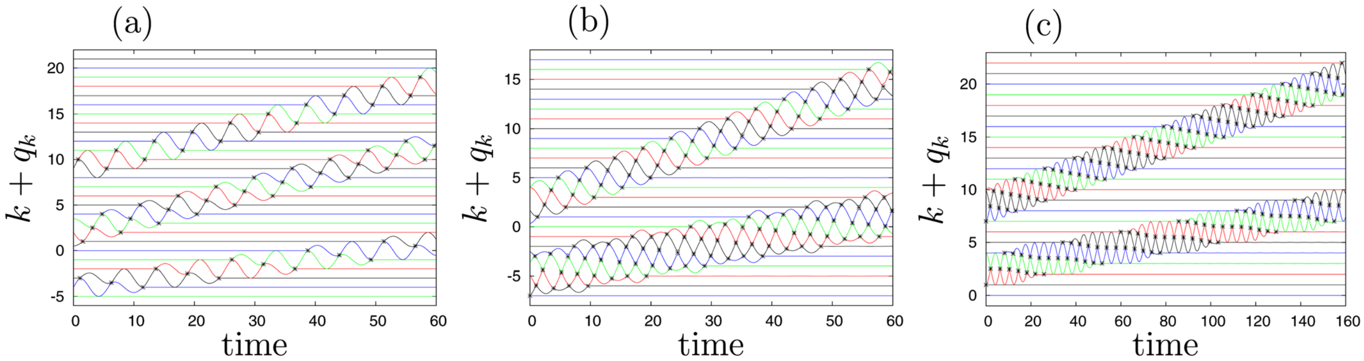

FIG. 1. In (a) we show two Robnik-Prosen compactons (upper and lower pulses) with $n = 1, 2$, together with the only one two-particle compacton, for which we have an analytic expression $-q_{-1} = q_1 = 0.5, p_1 = -1, q_0 = p_{-1} = p_1 = 0$. All other multi-particle compactons shown in (b),(c) have been found numerically. Lines are trajectories of the particles, markers depict collisions.

The simplicity of the dynamics allows one a very efficient integration strategy of this nonlinear oscillator chain, at which the collision times are determined explicitly by using standard inverse trigonometric functions only. This allows one to proceed to very large times with the accuracy of the double precision numerical arithmetic. Contrary to previous studies of the model (1), where the focus of the interest was on the heat conductivity properties of the lattice attached to thermostats,[8,22–24] we are interested in the *energy spreading* problem. We assume that initially only a small localized part of the lattice is excited, while all other oscillators are at rest, and characterize the spreading of the energy from such an initial configuration. We will see that the main ingredients of the dynamics are compactons and chaotic breathers, which we discuss in Sec. III.

III. COMPACTONS AND CHAOTIC STATES

A. Examples of compactons

The notion of compactons as of solitary waves with a compact support in nonlinear systems has been introduced by Rosenau.[25–27] Such compact solutions exist in strongly nonlinear partial differential equations. In strongly nonlinear lattices, the corresponding traveling waves and breathers are not strictly compact but have superexponentially fast decaying tails.[28–32] Remarkably, in the Ding-Dong model (1) strictly compact traveling waves exist. One family of such waves has been found by Prosen and Robnik,[8] we call these one-particle compactons, as these are solutions where at some moment of time just one oscillator in the lattice is excited (i.e., has non-zero energy). The family is determined by the coordinate and the momentum at the excited site: $q = 0, p = [\cos(\pi/(2(n+1)))]^{-1}$.

We have found several other compactons, which are many-particle pulses, i.e., at each moment of time at least several oscillators are excited (only for one of these waves we have found an analytical formula, all other are obtained numerically). These compactons, together with two representatives of the Prosen-Robnik family, are presented in Fig. 1.

B. Stability of compactons

One-particle compactons possess a remarkable one-side stability. We illustrate this in Fig. 2, where we show the evolution for a slightly disturbed compacton with $n = 1$ (it has unit energy) from the Prosen-Robnik family. For initial energies slightly less than 1, the compacton survives for a very long time, even if the perturbation is relatively large (Fig. 2(a)), while for energies slightly larger than $E = 1$ its life time is very short (Fig. 2(b)).

To understand the one-side stability, we apply a perturbative approach. Suppose that the initial energy of particle 0 is $E_0 = 1 - \varepsilon_0, \varepsilon_0 \ll 1$. Then at the next collision, the particle 1 will get velocity $V_1 = \sqrt{2E_0 - 1}$. After some time t, defined from the equation $V_1 \sin t - \cos t + 1 = 0$, particles 0 and 1 will collide again, and after this collision the particle 0 will have energy $\tilde{E}_0 = E_0 \cos^2 t$. This energy is "lost" due to the perturbation. Determining t to the leading order in ϵ_0, we find $\tilde{E}_0 = \varepsilon_0^2$. Thus, particle 1 will have energy

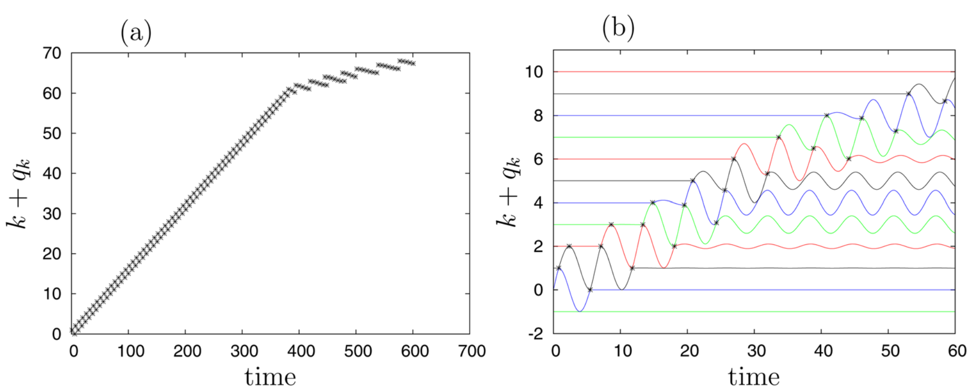

FIG. 2. Evolution of the perturbed compacton (cf. top pulse in Fig. 1(a)). (a): relatively large negative perturbation $\varepsilon_0 = 0.016$, (b): small positive perturbation $\epsilon_0 = -10^{-10}$.

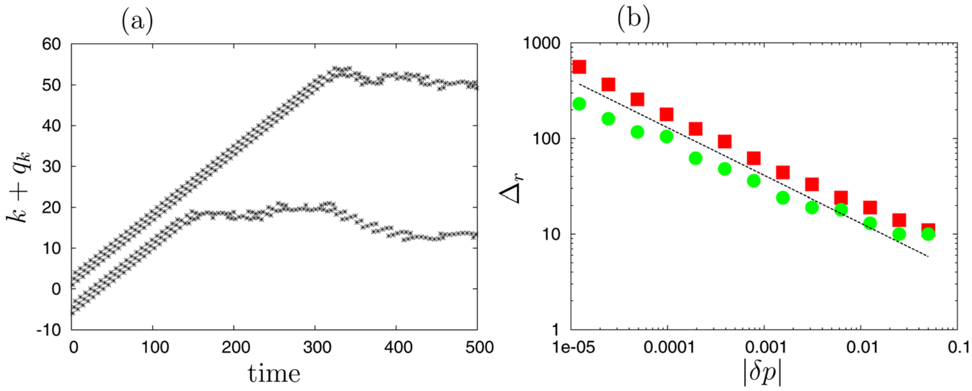

FIG. 3. (a) Evolutions of a perturbed two-site compacton (middle pulse in Fig. 1(a)) for $\delta p = -10^{-3}$ (upper case) and $\delta p = 10^{-3}$ (bottom case). (b) Maximal propagation of a compacton in dependence on δp (squares negative δp, circles positive δp; the line has slope -0.5).

$E_1 = 1 - \varepsilon_0 - \varepsilon_0^2$, and the whole cycle repeats. Summarizing, we obtain the following equation for the energy losses:

$$\varepsilon_{n+1} = \varepsilon_n + \varepsilon_n^2. \tag{2}$$

Approximating the evolution as a continuous one by replacing $\varepsilon_{n+1} - \varepsilon_n \longrightarrow \frac{d\varepsilon}{dn}$ we obtain that the losses grow as

$$\varepsilon_n = \frac{\varepsilon_0}{1 - n\varepsilon_0}. \tag{3}$$

Thus, the life time of the compacton (in propagation sites) is $\sim (\varepsilon_0)^{-1}$.

The life time for the positive perturbations ($\varepsilon_0 < 0$) is extremely small, so that we could not construct a perturbation theory and find an expression for it. One can see from Fig. 2(b) that for $\varepsilon_0 = -10^{-10}$ the life time of a compacton is around 10.

The simplest multi-site compacton shown in Fig. 1(a) is less stable to perturbations than the one-particle compacton, but its stability is relatively symmetric to the sign of the perturbation. We illustrate this in Fig. 3. There we perturb the initial state of the compacton as $q(-1) = -q(1) = -0.5, p(0) = -1 + \delta p$, and all other initial momenta and positions are zero. While the case of positive δp is slightly more unstable than that of negative perturbations, the maximal propagation length of the compacton Δ_r (defined as the position of the maximal site having non-zero energy) in both cases scales as $\sim (|\delta p|)^{-1/2}$.

C. Chaotic states

As already discussed in the previous studies of the Ding-Dong model,[8,23] general dynamical regimes are typically chaotic. A minimal chaotic state must include three particles (because of the additional conservation law, for a more general non-symmetric situation chaos is possible for two colliding oscillators[33]), thus two-particle regimes are periodic or quasiperiodic. We illustrate chaotic and quasiperiodic regimes in a 3-particle-lattice in Fig. 4. Here, we show a Poincaré map, depicting the coordinate and momentum of the left particle at the time instants when the central one and the right one collide. Remarkably, for $E_{cm} = 0$, we have found that for all tested initial conditions the particles neighboring to the excited three ones remain untouched if the total energy of three initially excited particles is less than 0.75. This is the case of a perfectly chaotic breather that does not spread.

IV. SPREADING OF INITIALLY LOCALIZED FIELD

In this section, we consider properties of spreading from rather general initial conditions. We first consider simple initial states, where the initial perturbation is restricted to one particle, initial energy E of which is varied in a wide range. For this initial state, we calculated the maximal range of propagation Δ_r up to time $t_{end} = 10^4$ and plotted this range vs the initial energy in Fig. 5. One can see "resonances" due to closeness to the compactons. The main resonance at $E = 1$ in panel (a) is due to the main compacton ($n = 1$) from the Prosen-Robnik family. There are many other resonances, as after several collisions the energy of the right-most particle becomes close to the value for one of the compactons, and such a perturbation propagates to a large distance. Formally, the resonances can be infinitely high (if perfect compactons are created) but we do not see this because of a finite resolution of initial energies.

In our next setup, we prepared a random initial state, setting velocities of several neighboring particles to random

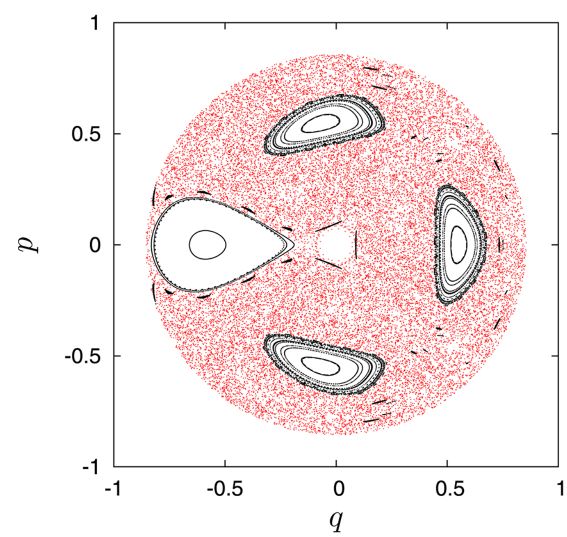

FIG. 4. Coordinates and momenta of the left particle at moments of collision of the central and right ones. Grey dots correspond to chaotic orbits, black dots to quasiperiodic ones. Total energy is $E = 0.8$, $E_{cm} = 0$.

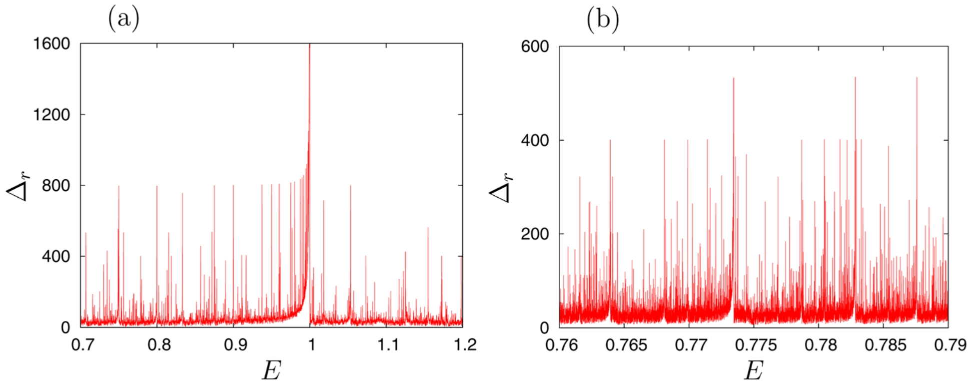

FIG. 5. Range of propagation from the one-site excitation, in dependence on the initial energy of the particle E. Right panel is the enlarged section showing the finer structure of resonances.

numbers. A typical evolution is shown in Fig. 6. Panel (a) shows the space-time pattern of collisions. One can see here the characteristic structures we typically observed. A compact-like wave first propagates to the right, close to site 30 due to losses of energy it transforms to another quasi-compacton, which propagates until site 80 where it gets destroyed and gives rise to an irregular collision pattern. At sites 9 and 10 a regular breather is created, the pattern of collisions of these two particles is quasiperiodic (see panel (c)). At the left part of the space-time diagram, one can see a chaotic quasi-breather that includes particles -20,-19, -18, and -17. Particle -21 is not excited during the whole time interval presented, but particles in the quasi-breather exchange energy in an erratic manner (see panel (b)). We cannot exclude that in course of time this breather will spread or, probably, shift irregularly along the lattice performing a random walk. In fact, one can interpret the pattern seen at sites $70 < q < 90$ at times $t > 500$ as such a randomly moving breather ("chaotic spot"). We stress here that the "empty places" in Fig. 6(a) have in fact non-zero energy (if they are between the left-most and the right-most sites), but this energy is insufficient for collisions (cf. panel (c)).

We discuss now some general properties of the spreading process, basing on the reversibility of the dynamics. Energy cannot spread in such a way that the collisions disappear, as this would contradict reversibility. If there has been some initial spreading stage, the final state cannot consist of quasiperiodic and periodic breathers only, as such a state being reversed in time would propagate backward to infinity. Thus, the final state is either a perfect combination of breathers and exact compactons or has at least one chaotic component. Typically, one observes several "active" spots where collisions occur and which are chaotic or quasiperiodic, while the rest of the energy is spread along the non-active sites that do not collide with neighbors. There is no limit for the maximal distance in space between the right-most and the left-most sites, but energy cannot be uniformly distributed along the lattice because in this case collisions would disappear.

To characterize the spreading, we calculated the spatial entropy of the energy distribution, according to the standard Boltzmann definition $I = -\sum_k \varepsilon_k \log\varepsilon_k$, where $\varepsilon_k = E_k/\hat{E}$ is the energy at site k normalized by the total energy E. Starting from different random initial conditions, we calculated the entropies of the configurations at different times, and than averaged them over the ensemble. The results presented in Fig. 7 show that the spreading stops at some maximal entropy level, which grows with the total energy as shown in panel (b).

V. DISORDERED DING-DONG MODEL

In this section, we study a disordered Ding-Dong model. Disorder can be introduced in model (1) in three ways: disorder in the spacings between the oscillators, disorder in the particle masses, and disorder in the oscillator frequencies. The latter situation is very difficult for numerical modeling, as there is no easy way to calculate the collision time of two

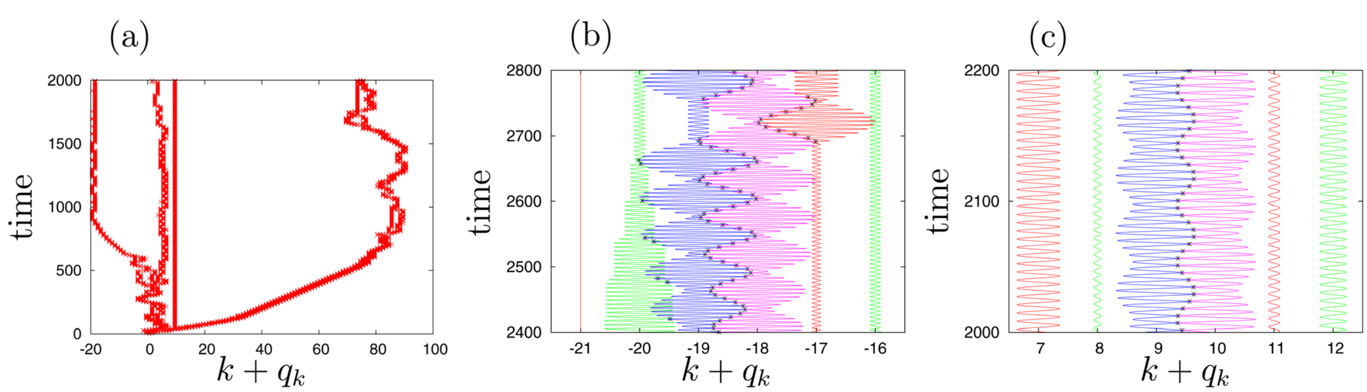

FIG. 6. Evolution from a particular realization of random initial momenta (5 sites are excited with total energy $E = 5$). Panel (a) shows the pattern of collisions, in panels (b) and (c) we show collisions (with markers) and particle trajectories (with lines) at a chaotic spot and at a quasiperiodic breather, respectively (see text for details).

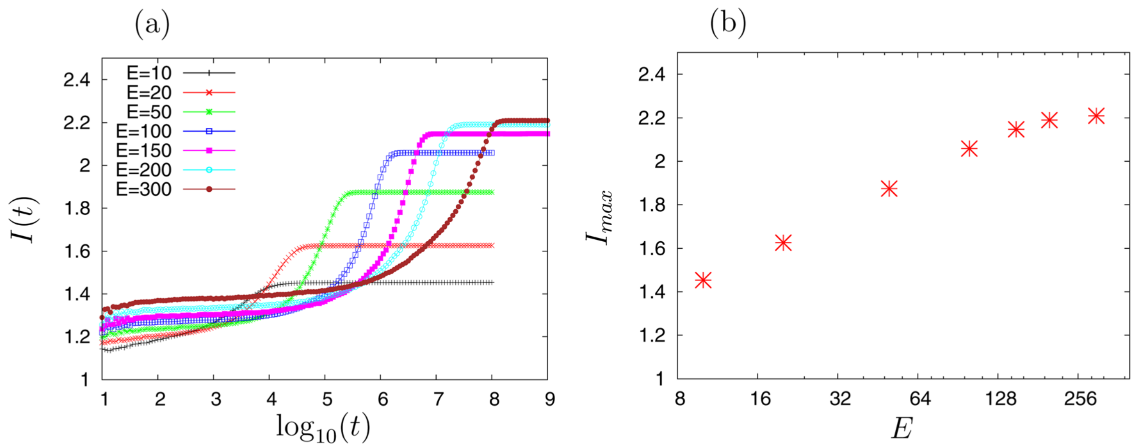

FIG. 7. (a): Evolution of the entropy of the energy distribution, starting from random initial conditions (momenta of particles with $-10 \le k \le 10$ are random Gaussian variables), for different total energies. (b): Dependence of the maximal entropy on the total energy.

harmonic oscillators with incommensurate frequencies. Thus, we restrict our attention to the first two cases. In the case of disorder in distances between the oscillators, the Hamiltonian is still (1), but the collision condition is now $q_k = r_{k,k+1} + q_{k+1}$ with random spacings $r_{k,k+1}$. At a collision, the momenta are exchanged $p_k \to p_{k+1}, p_{k+1} \to p_k$. In the case of mass disorder, we write the Hamiltonian as

$$H = \sum_k \left(\frac{p_k^2}{2m_k} + \frac{q_k^2}{2m_k} \right), \tag{4}$$

so that all the frequencies are one. The collision condition is $q_k = 1 + q_{k+1}$, but the momenta are exchanged according to

$$p_k \to \frac{2m_k p_{k+1} + (m_k - m_{k+1})p_k}{m_k + m_{k+1}},$$
$$p_{k+1} \to \frac{2m_{k+1} p_k - (m_k - m_{k+1})p_{k+1}}{m_k + m_{k+1}}. \tag{5}$$

In both cases, we studied chains with the spacings or the masses independent and uniformly distributed in $1 - \delta r < r_{k,k+1} < 1 + \delta r, 1 - \delta m < m_k < 1 + \delta m$.

The main effect of disorder is that the compactons disappear, because the translational invariance is broken. On the other hand, chaotic and quasiperiodic breathers may exist. We have determined spreading properties for different realizations of disorder and for different initial conditions, and in all cases have found that after some initial stage the spreading eventually stops and the maximal spreading range $\Delta = k_{max} - k_{min}$ remains constant. Here, k_{max} and k_{min} are the indices of the right-most and the left-most excited sites, respectively. Of course, this conclusion is based on numerical calculations within a finite time interval. To be more precise, the conclusion on "non-spreading" was made after there was no any spreading event (i.e., the left-most and the right-most excited sites remained unchanged) during the time interval 10^{10}. In Fig. 8, we present the statistics of the spreading ranges for different types of disorder, different total energies E, and different disorder strengths. One can see that the maximal spreading range grows as a power law of the total energy with power ≈ 0.7. The spreading range practically does not depend on the disorder level.

In some cases, the initial configuration was not chaotic, in this case, no spreading was observed (thus the minimal spreading interval is 10 for energy $E = 2$), in all other cases,

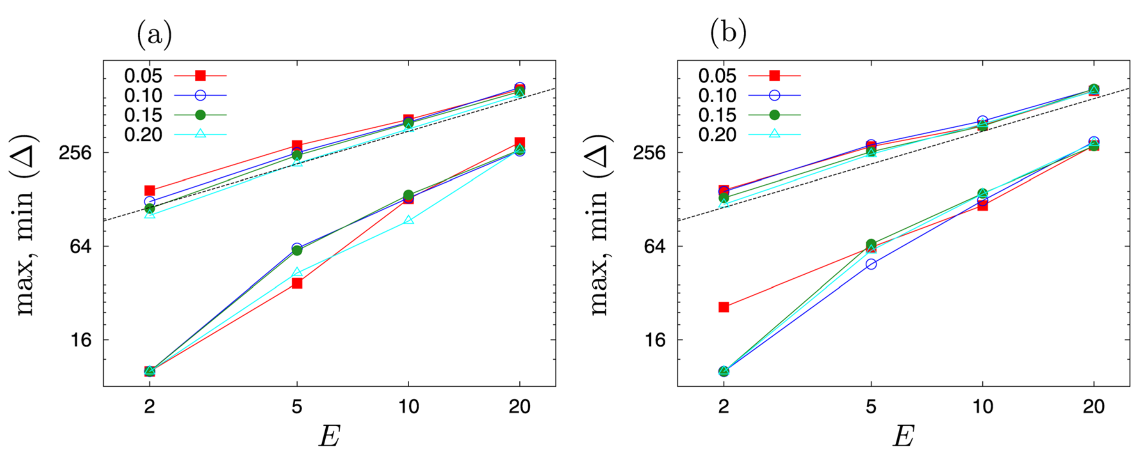

FIG. 8. Maximal and minimal spreading ranges Δ vs total energy, determined from ensembles of approximately 2000 realizations of disorder and initial conditions (initially 10 sites were excited randomly). (a): disorder in distances, (b): disorder in masses. Values of δr and δm are shown on the panels. Dashed lines correspond to $\Delta \sim E^{0.7}$.

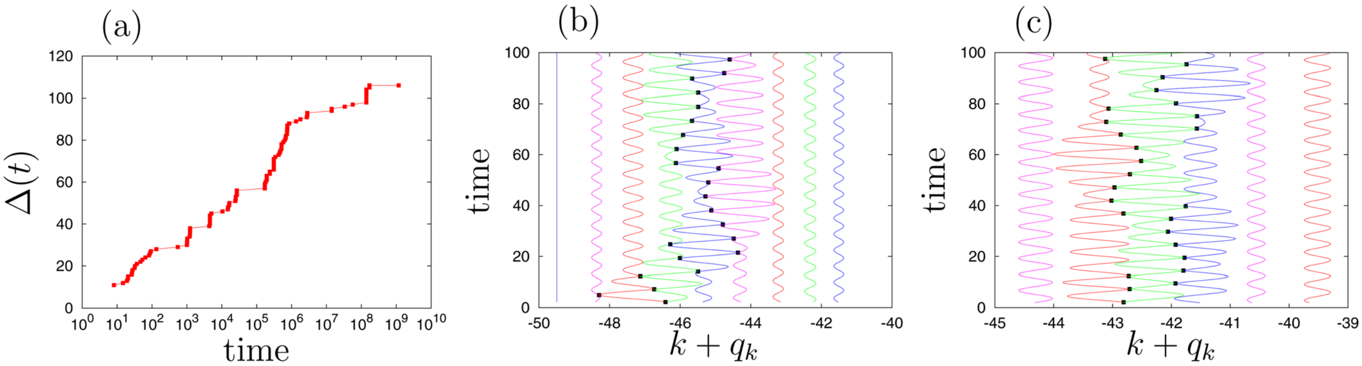

FIG. 9. (a): Evolution of the spreading range Δ for a particular configuration of the distance disorder, for $\delta r = 0.2$ and $E = 5$. Panels (b) and (c) show a chaotic spot near the left edge of the lattice, just after the last spreading event has happened, and after time interval 10^9, during which Δ remains constant, respectively. One can see that the spot is slightly shifted. Time in (b) and (c) is marked relative to the beginning of the corresponding intervals.

the final state consists of a few chaotic spots. We illustrate this in Fig. 9. Here, for one configuration of disorder and for randomly chosen initial conditions we show in panel (a) how the energy spreads along the lattice. The maximal spreading range $\Delta = 106$ is reached at $t \approx 1.7 \cdot 10^8$. After this time, the activity (i.e., collisions) is observed at two spots which very slowly shift along the lattice.

VI. CONCLUSIONS

In this paper, we have studied properties of energy spreading in a nonlinear Ding-Dong model. This nonlinear lattice is not generic, as it has a strict nonlinearity threshold, below which the oscillations are purely linear and do not interact. We have described two elementary excitations in the lattice—compactons and chaotic breathers. Compactons are relatively stable objects, so quite often we observe appearance of quasi-compactons with a long lifetime from general initial conditions. Due to this, initially localized energy is spread over a large domain, but the final stage is highly nonuniform: while at many places energy is below the collision threshold, there exist several spots—typically regular or chaotic breathers. Correspondingly, the spatial entropy of the energy distribution saturates at some level depending on the total energy of the lattice. Furthermore, we studied a disordered Ding-Dong model. Here, compactons do not exist and the energy spreading is finite: the field remains localized up to maximal times of our calculations. Again, several chaotic spots are formed together with a subthreshold background.

It is instructive to compare the properties of the Ding-Dong model with those of lattices with smooth potentials. In the latter case, one also observes compactons (while with superexponential tails[32]). Exact chaotic breathers appear to be impossible, because the interaction with the neighbors has no threshold and the neighbors would be excited by the noisy oscillations of such a breather. Similar to the Ding-Dong case, disorder destroys compactons, and the energy spreading is a slow subdiffusive process. In a disordered smooth nonlinear lattice so far no stop of energy spreading has been observed, although there are indications that the spreading slows down with the time.[20,21] Also, the scaling properties of the largest Lyapunov exponent in smooth lattices[34] suggest that chaos might extinct at the late stages of spreading. The eventual stop of spreading in the Ding-Dong model is, to the best of our knowledge, the first observation of the localized but chaotic dynamics in nonlinear lattices. However, nonlinear disordered lattices with analytic interaction potentials possess a property that every lattice site is coupled (although indirectly) with all other sites, so that chaotic spots influence the whole lattice. This feature implicates that systems with analytic and non-analytic interaction potentials might demonstrate different asymptotic scenaria. It would be very interesting to investigate smooth lattices close to the Ding-Dong model and to compare their localization properties (similar to the studies of smooth approximations of scattering billiards[35]).

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