Self-Organized Quasiperiodicity in Oscillator Ensembles with Global Nonlinear Coupling

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We describe a transition from fully synchronous periodic oscillations to partially synchronous quasiperiodic dynamics in ensembles of identical oscillators with all-to-all coupling that nonlinearly depends on the generalized order parameters. We present an analytically solvable model that predicts a regime where the mean field does not entrain individual oscillators, but has a frequency incommensurate to theirs. The self-organized onset of quasiperiodicity is illustrated with Landau-Stuart oscillators and a Josephson junction array with a nonlinear coupling.

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Ensembles of globally (all-to-all) coupled oscillators are popular models of physical, biological, and social phenomena. The main effect observed in these models is the collective synchrony, when a large part or all units adjust their rhythms and produce a nonzero mean field, which has the same frequency as the synchronized majority (see experiments [1]). In the simplest setup, this state appears from the fully asynchronous one via the Kuramoto transition [2,3]. However, complexity of individual oscillators and/or of the coupling function can result in interesting dynamics like chaos of the mean field, clustering and multistability, splay states, etc. [4]. In this Letter, we demonstrate a novel transition from full synchrony of identical, globally coupled periodic oscillators to a quasiperiodic regime, when oscillators are not locked to the periodic mean field they produce, but remain, however, coherent (cf. [5]). Such counter-intuitive, partially synchronous states have been observed by van Vreeswijk [6] and later in [7]. Here, we demonstrate for the first time that these states naturally appear when the coupling between oscillators nonlinearly depends on the order parameters.

We start with an analytical description of the transition from full to partial synchronization in the simplest setup. Let us first recall the Kuramoto-Daido [2,8] model of N globally coupled identical phase oscillators

$$\dot{\theta}_k = \omega + N^{-1}\sum_{j=1}^{N} h(\theta_j - \theta_k). \tag{1}$$

Here, $h(\theta) = \sum_n h_n e^{in\theta}$ is a general coupling function. Equation (1) is already averaged over period $2\pi/\omega$ of fast oscillations; thus, h depends on phase differences only. With the help of the generalized order parameters [8] (which vanish in the asynchronous state and are nonzero beyond the transition to synchrony),

$$Z_n = \frac{1}{N}\sum_{j=1}^{N} e^{in\theta_j} = \langle e^{in\theta}\rangle \tag{2}$$

system (1) can be rewritten as

$$\dot{\theta}_k = F(\theta_k) = \omega + \sum_n h_n Z_n e^{-in\theta_k}. \tag{3}$$

As $h_n = \text{const}$, the coupling function F linearly depends on the order parameters Z_n. We generalize model (3) by allowing a nonlinear dependence of the coupling function F on the order parameters Z_n:

$$F(\theta) = \omega + \sum_n h_n^{(1)} Z_n e^{-in\theta} + \sum_{n,l} h_{nl}^{(2)} Z_n Z_l e^{-i(n+l)\theta} + \sum_{n,l,m} h_{nlm}^{(3)} Z_n Z_l Z_m e^{-i(n+l+m)\theta} + \ldots. \tag{4}$$

Though we remain in the framework of phase approximation, this generalization allows us to account for the dependence of the form of the coupling on its magnitude. Like in the linear case, $F(\theta)$ contains only slow, in comparison to the oscillation frequency ω, terms.

We first analyze a particular but completely solvable example of nonlinear coupling, where $F(\theta)$ depends only on the principal order parameters $Z_{\pm 1}$ and contains only the harmonics $e^{\pm i\theta}$ [9]. This means that nonlinear terms $\sim Z_1$, $\sim |Z_1|^2 Z_1$, $\sim |Z_1|^4 Z_1, \ldots$ in (4) can be gathered as $H(|Z_1|^2, \varepsilon) Z_1 e^{-i\theta}$ + c.c., where we have also introduced a bifurcation parameter ε. Nonoscillatory even terms $\sim |Z_1|^2, \ldots$ can be absorbed in frequency, $\omega \to \omega(K)$. Denoting $Z_1 = Ke^{i\Theta}$ and $H = -iR(K,\varepsilon)e^{i\beta(K,\varepsilon)}/2$, we obtain

$$\dot{\theta}_k = \omega(K) + R(K,\varepsilon)K\sin[\beta(K,\varepsilon) + \Theta - \theta_k]. \tag{5}$$

It follows that the regime of full synchrony, $\theta_1 = \ldots = \theta_N = \Theta$ and $K = 1$, has frequency $\Omega = \omega(1) + R(1,\varepsilon) \times \sin\beta(1,\varepsilon)$. This regime is stable if $d\dot{\theta}_k/d\theta_k = -R\cos\beta(1,\varepsilon) < 0$, i.e., if $-\pi/2 < \beta(1,\varepsilon) < \pi/2$. Thus, at two critical values of the bifurcation parameter, determined by $\beta(1,\varepsilon_q) = \pm\pi/2$, the full synchrony breaks. Now we show that beyond these points, there exists a periodic mean field with $0 < K < 1$ and some frequency $\Omega = \dot{\Theta} \neq \omega_{\text{osc}} = \langle\dot{\theta}\rangle$. This field does not entrain the oscillators, and therefore the quasiperiodic regimes are observed. For the phase differences $\psi_k = \theta_k - \Theta$, we write

$$\dot{\psi}_k = \omega(K) - \Omega + R(K,\varepsilon)K\sin[\beta(K,\varepsilon) - \psi_k]. \quad (6)$$

These equations should be complemented by the expression for the real amplitude of the order parameter $K = \langle e^{i\psi}\rangle$, which follows from the definition (2). Considering the thermodynamic limit $N \to \infty$ and replacing summation by integration (cf. [2]), we write the probability to observe a certain value of ψ as $\rho(\psi) \sim |\dot{\psi}|^{-1}$, which yields after normalization

$$\rho(\psi) = \frac{\sqrt{[\omega(K) - \Omega]^2 - R^2K^2}}{2\pi|\omega(K) - \Omega + RK\sin(\beta - \psi)|}. \quad (7)$$

Using this distribution, we obtain a self-consistency condition $K = \int_{-\pi}^{\pi} e^{i\psi}\rho(\psi)d\psi$ for determination of unknown K and Ω. Substituting (7) in this integral, we get

$$K = \int_{-\pi}^{\pi} d\psi \frac{-ie^{i\beta}\cos\psi\sqrt{[\omega(K) - \Omega]^2 - R^2K^2}}{2\pi|\omega(K) - \Omega + RK\cos\psi|}.$$

The imaginary part yields $\cos\beta = 0$, while the real part of this equation gives

$$K = (RK)^{-1}\{|\omega(K) - \Omega| - \sqrt{[\omega(K) - \Omega]^2 - R^2K^2}\}.$$

The final expressions for the order parameter K and frequency Ω are

$$\beta(K,\varepsilon) = \pm\pi/2; \qquad \Omega = \omega(K) \pm R(K,\varepsilon)(1 + K^2)/2. \quad (8)$$

The magnitude K of the order parameter is solely determined by the nonlinear phase shift β. Remarkably, K takes exactly the value that keeps this phase shift at its critical value: $\beta(K,\varepsilon) = \beta(1,\varepsilon_q) = \pm\pi/2$. Thus, when the full synchrony is lost, the ensemble of oscillators organizes itself in a way that the phase of the mean field that forces an individual oscillator is exactly on the stability border. This is similar to the self-organized criticality [10], where the system operates at a critical state in a self-adjusted manner. The frequencies of the mean field Ω [Eq. (8)] and of each oscillator [the latter is obtained by integrating Eq. (6)] $\omega_{\rm osc} = \omega \pm RK^2$ are determined by the coupling magnitude R and depend on the bifurcation parameter ε in a smooth way. Hence, generally one observes a quasiperiodicity: frequencies Ω and $\omega_{\rm osc}$ are incommensurate. Being considered as a critical phenomenon, the self-organized quasiperiodicity can be characterized by trivial critical indices equal to one. Indeed, because $\beta(K,\varepsilon)$ is generally a smooth function of both arguments, near the criticality $\beta(K,\varepsilon) \approx \beta(1,\varepsilon_q) + \beta_K(K-1) + \beta_\varepsilon(\varepsilon - \varepsilon_q)$. Therefore, $K \approx 1 - (\beta_\varepsilon/\beta_K)(\varepsilon - \varepsilon_q)$, and, similarly $\Omega - \omega_{\rm osc} \propto (\varepsilon - \varepsilon_q)$.

As the first example for the application of our theory, we consider an ensemble of N identical Landau-Stuart oscillators (variables A_k), coupled via a common nonlinear load (variable B). Examples of systems with common load coupling include contractile elements attached to a damped mass-spring oscillator in a model of a muscle [7], pedestrians on a bridge [11], etc., Our model reads

$$\dot{A}_k = (1 + i\omega_a)A_k - |A_k|^2A_k + e^{i\xi}B, \quad (9)$$

$$\dot{B} = -\gamma B + i\omega_b B + i\eta|B|^2B + \frac{\varepsilon}{N}\sum_{k=1}^{N} A_k. \quad (10)$$

To treat the system, we assume that the mean field is periodic with the (yet unknown) frequency $\Omega = \dot{\Theta}$. For small forcing B, we can use phase approximation, assuming $|A_k| = 1$. Allowing oscillators to have different, slowly varying phases ψ_k, we write $A_k = e^{i\theta_k} = e^{i[\Theta + \psi_k(t)]}$. Hence, $Z_1 = \langle A_k\rangle = e^{i\Theta}\langle e^{i\psi}\rangle = Ke^{i\Theta}$. We seek now for a periodic, with frequency Ω, solution of (10): $B = be^{i\Theta}$. Then b obeys $\gamma b + i(\Omega - \omega_b)b - i\eta|b|^2b = \varepsilon K$. Assuming that the damping is large compared to the frequency mismatch, $\gamma \gg |\Omega - \omega_b|$ (what also ensures enslaving of B by A_k), we can solve this equation up to the third order in ε as $b \approx \varepsilon K\gamma^{-1}\exp(i\eta\gamma^{-3}\varepsilon^2K^2)$. Substituting this in (9) and extracting the equation for the phase ψ_k, we obtain

$$\dot{\psi}_k = \omega_a - \Omega + \varepsilon K\gamma^{-1}\sin(\xi + \eta\gamma^{-3}\varepsilon^2K^2 - \psi_k), \quad (11)$$

i.e., a particular case of the model (6) with $R = \varepsilon\gamma^{-1}$ and $\beta = \xi + \eta\gamma^{-3}\varepsilon^2K^2$. Hence, the critical coupling $\varepsilon_q^2 = \eta^{-1}\gamma^3(\pm\pi/2 - \xi)$, $K = \varepsilon_q/\varepsilon$ and $\omega_{\rm osc}$ and Ω are readily obtained. In Fig. 1, we compare the theory with the simulations of Eqs. (9) and (10) for $\omega_a = \omega_b = 1$, $\gamma = 5$, $\eta = 10^3$, $\xi = 0.475\pi$, and nearly uniform initial distribution of θ_k. As expected, for small coupling, the identical oscillators are fully synchronous, but for $\varepsilon > \varepsilon_q \approx 0.1088$ (approximate formulae above give $\varepsilon_q = 0.099$), the full synchrony is lost, what is reflected in the decrease of the

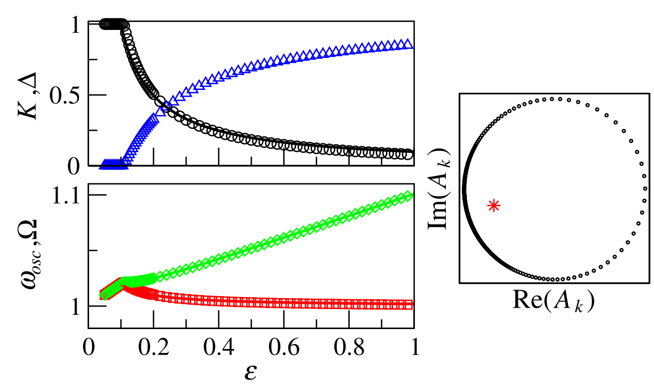

FIG. 1 (color online). Simulation of Eqs. (9) and (10). Left panels: Order parameter K (circles), frequencies of oscillator $\omega_{\rm osc}$ (squares) and of the mean field Ω (diamonds) vs coupling strength ε. Corresponding theoretical curves are shown by solid lines and nearly coincide with numerics. Normalized minimal distance Δ (triangles) demonstrates the absence of clusters in the quasiperiodic regime. Right panel shows a snapshot of the ensemble (circles), which can be considered as a sampled distribution (7); star is the mean field.

order parameter. The distribution of phases θ_k, which is δ-function in the synchronous state, now broadens so that the phases are nonuniformly distributed around the limit cycle cf. (7). The larger $\varepsilon - \varepsilon_q$, the less pronounced is the maximum of the distribution (and, hence, K). The maximum of this distribution rotates with a frequency different from that of individual units; this is a manifestation of the difference of Ω and ω_{osc}.

Let us use Eq. (11) to exemplify the mechanism of self-organization in the ensemble. Because the phase shift $\beta = \xi + \eta\gamma^{-3}\varepsilon^2K^2$ varies with ε, the interaction of synchronized ($K = 1$) oscillators changes from attractive one for small ε to repulsive one for large ε. However, for $\varepsilon > \varepsilon_q$, the system cannot desynchronize completely, as for $K = 0$, synchrony is stable. As a result, the system "chooses" such a value $0 < K < 1$ that $\varepsilon^2K^2 = \varepsilon_q^2$ and $\beta = \pi/2$, and thus settles exactly at the border between attraction and repulsion.

Typically, if periodic oscillators are subject to a periodic force, different phase-locking phenomena may be observed. In the case of self-organized quasiperiodicity, we have not observed phase-locked states in model (9) and (10). Indeed, a stable phase locking of identical oscillators would mean formation of clusters, where the states of different oscillators coincide. To show the absence of clusters, we compute the minimal distance (in the state space) between the oscillators, normalized by the ensemble size: $\Delta = (N/2\pi)\min(|A_k - A_j|)$ (the minimum is taken over all pairs and over a long run). We show in Fig. 1 that when the stability of the synchronous one-cluster solution is lost, $\Delta > 0$, i.e., no clusters exist.

For the next example, we recall a well-studied model of the array of Josephson junctions (their capacitances are neglected), shunted by a common RLC load [12],

$$\frac{\hbar}{2er}\frac{d\Psi_k}{dt} + I_c\sin\Psi_k = I - \frac{dQ}{dt}, \tag{12}$$

$$\frac{d\Phi}{dt} + R\frac{dQ}{dt} + \frac{Q}{C} = \frac{\hbar}{2e}\sum_k\frac{d\Psi_k}{dt}, \tag{13}$$

where we use the same notations as in [12]. For a linear RLC load and weak coupling, Eqs. (12) and (13) can be reduced to Eq. (1) with $h(\cdot) \sim \sin(\cdot)$, i.e., to the Kuramoto model [12]. Below, we analyze nonlinearly coupled junctions and demonstrate that their phase dynamics can be described by the general phase model (4). Namely, we consider a nonlinear inductance for which the magnetic flux Φ depends on the current $\dot{Q}$ as $\Phi = L_0\dot{Q} + L_1\dot{Q}^3$. Using a transformation [12]

$$\theta_k = 2\arctan\left[\sqrt{\frac{I - I_c}{I + I_c}}\tan\left(\frac{\Psi_k}{2} + \frac{\pi}{4}\right)\right] - \Omega t,$$

we rewrite the system as

$$\frac{d\theta_k}{dt} = \omega - \Omega - \frac{\omega}{I^2 - I_c^2}\frac{dQ}{dt}[I - I_c\cos(\Omega t + \theta_k)], \tag{14}$$

$$\frac{d^2Q}{dt^2} + \frac{R + Nr}{L_0}\frac{dQ}{dt} + \frac{Q}{L_0C} + \frac{L_1}{L_0}\frac{d}{dt}\dot{Q}^3 = rL_0^{-1}(I^2 - I_c^2)\sum_k[I - I_c\cos(\Omega t + \theta_k)]^{-1}, \tag{15}$$

where $\omega = 2er\hbar^{-1}\sqrt{I^2 - I_c^2}$ is the frequency of uncoupled junctions. The right hand side of (15) can be represented as a Fourier series, where the components are proportional to the generalized order parameters $Z_n = \langle e^{in\theta}\rangle$:

$$NrL_0^{-1}\sqrt{I^2 - I_c^2}\sum_n Z_n e^{in\Omega t}I_c^{-n}(I - \sqrt{I^2 - I_c^2})^n. \tag{16}$$

Representing Q as a Fourier series $Q = \sum_n q_n e^{in\Omega t}$, substituting this in (14), and averaging over the period $2\pi/\Omega$, we find that the forcing in the phase equation is proportional to the principal Fourier components $q_{\pm 1}$ only

$$\frac{d\theta_k}{dt} = \omega - \Omega - \frac{i\omega I_c\Omega(-q_{-1}e^{i\theta_k} + q_1e^{-i\theta_k})}{2(I^2 - I_c^2)}. \tag{17}$$

Because of the nonlinearity of the load (15), these Fourier components depend on all generalized order parameters Z_n. Hence, Eq. (17) is a particular case of general nonlinear coupling of type (4). We cannot solve Eq. (17) analytically, but numerical simulations of an array of $N = 100$ junctions demonstrate a loss of full synchrony via the transition to quasiperiodicity, as shown in Fig. 2. While in the regime of full synchrony, the dc voltage V_{dc} on each junction is determined by the oscillation frequency of the load: $2eV_{dc} = \hbar\langle\dot{\Psi}_k\rangle = \hbar\Omega$. In the nonsynchronized state, the ratio $\frac{2eV_{dc}}{\hbar\Omega}$ is in general irrational.

Above, we have illustrated by the theory and two examples a mechanism of appearance of a self-organized quasiperiodic regime in the framework of phase dynamics approximation. In a general context beyond this approximation, such regimes establish when the stability of full synchrony, upon a change of a bifurcation parameter, gets lost. This loss of full synchrony can be followed for general globally coupled ensembles of identical oscillators

$$\dot{\mathbf{x}}_k = \mathbf{F}(\mathbf{x}_k, \mathbf{y}, \mathbf{g}; \varepsilon), \qquad \dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}, \mathbf{g}; \varepsilon), \tag{18}$$

where variables $\mathbf{x}_k$ describe individual systems, $\mathbf{g}(\mathbf{x})$ are mean fields (ensemble averages of some observables), and $\mathbf{y}$ describe the dynamics of the coupling circuit.

In the case of full synchrony in system (18), all $\mathbf{x}_k = \mathbf{x}$ are equal, and the fields $\mathbf{g}$ are simple functions of them. Then the total system (18) reduces to a system of order $\dim(\mathbf{x}) + \dim(\mathbf{y})$. Suppose this system possesses a stable solution with period T. Following [13], we can now investigate its stability towards the break of synchrony by calculating the so-called "evaporation multipliers." These multipliers μ define the stability of the synchronous cluster with respect to "evaporation" of individual sys-

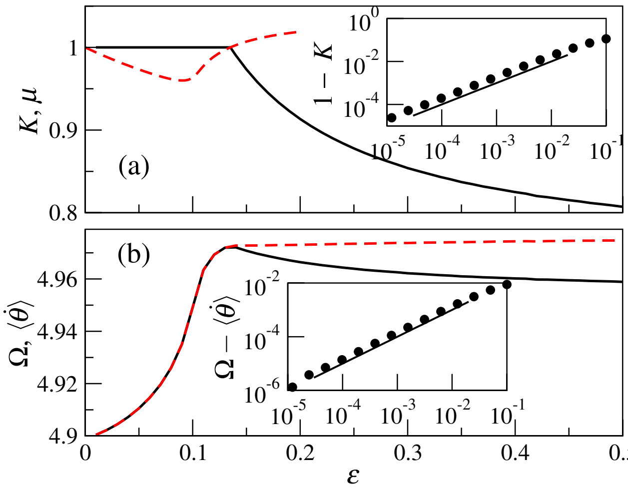

FIG. 2 (color online). Transition to self-organized quasiperiodicity in an array of nonlinearly coupled Josephson junctions (12) and (13). The order parameter $K = |Z_1| = |\langle e^{i\theta}\rangle|$ [(a), solid line] and the frequencies of the mean field [(b), dashed line] and of individual oscillators [(b) solid line] are shown as functions of dimensionless coupling strength $\varepsilon = N\hbar(2eI_cL_0)^{-1}$. Other dimensionless parameters are $I/I_c = 5$, $\hbar/[\sqrt{L_0C}(2erI_c)] = 4.8$, $L_1/(L_0I_c^2) = -0.5$, $2erRI_c/(L_0\hbar) = 0.05$. Insets demonstrate scaling $(1-K)$, $\Omega - \langle\dot{\theta}\rangle \propto \varepsilon - \varepsilon_q$. Stability of fully synchronous state is quantified by multiplier μ (see text), shown by dashed line in (a); transition to quasiperiodicity appears exactly when μ becomes large than one.

tems. They are obtained as eigenvalues of the mapping $\delta\mathbf{x}(\mathbf{t}) \rightarrow \delta\mathbf{x}(t+T)$, resulting from Eq. (18), linearized only with respect to the variables of individual oscillators: $\frac{d}{dt}\delta\mathbf{x} = \frac{\partial\mathbf{F}}{\partial\mathbf{x}}\delta\mathbf{x}$. In both above considered examples, the instability of one-cluster solution occurs as one real evaporation multiplier becomes larger than unity, with an excellent coincidence with the direct simulation (see Fig. 2).

We compare now our results with that of [6], where a transition from partial synchronization to asynchrony was considered for a particular model of integrate-and-fire oscillators with a retarded coupling. In this system, the regime of full synchrony is forbidden; thus, the instability of the asynchronous state leads to partial synchrony only. We consider a more general situation, where all regimes—full and partial synchrony, as well as asynchrony—are allowed, and partial synchrony appears via the self-organization due to nonlinearity in the coupling. (Such a state has been presumably observed in [7].)

In summary, we have studied ensembles with global coupling that nonlinearly depends on the order parameters. We have shown that such systems naturally exhibit partially synchronized quasiperiodic regimes, which exist in a very broad range of parameters. We have described analytically and illustrated numerically a mechanism of the self-organized quasiperiodicity; simulations show that these quasiperiodic regimes exist in case of nonidentical oscillators as well. We observed and analyzed analytically a novel transition from fully synchronous to partially coherent, quasiperiodic state. This bifurcation occurs when one real evaporation multiplier of the synchronous limit cycle solution becomes larger than 1. Generally, it is possible that a pair of complex multipliers crosses the unit circle. This case, where the transition cannot be described within the phase approximation but only in the framework of the general model (18), occurs, e.g., in globally coupled Hindmarsh-Rose neuronal models and will be analyzed elsewhere. Our findings show that accounting for a nonlinear coupling of oscillators is essential for understanding of the complex dynamics of globally coupled real-world systems, e.g., neurons.

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