Effects of Delayed Feedback on Kuramoto Transition

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We develop a weakly nonlinear theory of the Kuramoto transition in an ensemble of globally coupled oscillators in presence of additional time-delayed coupling terms. We show that a linear delayed feedback not only controls the transition point, but effectively changes the nonlinear terms near the transition. A purely nonlinear delayed coupling does not effect the transition point, but can reduce or enhance the amplitude of collective oscillations.

§1. Introduction

A transition to collective synchrony in an ensemble of globally coupled oscillators is known as the Kuramoto transition.[3] An important application of the theory is collective dynamics of neuronal populations. Indeed, synchronization of individual neurons is believed to play the crucial role in the emergence of pathological rhythmic brain activity in Parkinson's disease, essential tremor, and epilepsies; a detailed discussion of this topic and numerous citations can be found in Refs. 2), 4) and 8). One approach to suppress such an activity is to apply to the system a negative feedback loop.[5)–7)]

The goal of this paper is to develop a weakly nonlinear theory of the Kuramoto transition in the presence of linear and nonlinear time-delayed coupling terms. We heavily rely in our analysis on the corresponding treatment of the system without delay by Crawford.[1]

§2. From limit cycle systems to phase models

Here we introduce our basic model — an ensemble of autonomous oscillators subject to different types of global coupling. We take individual oscillators as Van der Pol ones and write the model as

$$\ddot{x}_i - \mu(1 - x_i^2)\dot{x}_i + \omega_i^2 x_i = 2\sqrt{2}\omega_i \xi_i(t) + \varepsilon' F(\overline{x}, \overline{y}), \tag{2·1}$$

where $\xi_i(t)$ is a δ-correlated Gaussian noise: $\langle \xi_i(t)\xi_j(t-t')\rangle = 2D\,\delta_{ij}\,\delta(t')$. The ensemble averages are defined as

$$\overline{x} = \frac{1}{N}\sum_{j=1}^{N} x_j\,, \qquad \overline{y} = \frac{1}{N}\sum_{j=1}^{N} \frac{\dot{x}_j}{\omega_j}\,.$$

In the reduction to phase equations we use the smallness of parameters μ and ε', and suppose the natural frequencies ω_i to be distributed in a relatively close vicinity of the mean frequency $\omega_0 \equiv N^{-1}\sum_{j=1}^{N}\omega_j$. Because $\mu \ll \omega_i$, the solution of the autonomous Van der Pol oscillator can be written as $x_i(t) \approx A_i(t)\cos(\varphi_i(t))$ where on the limit cycle $A_i \approx 2$ and $\dot{\varphi}_i = \omega_i$. Because $\varepsilon' \ll \mu$, coupling does not affect the amplitude (which remains ≈ 2), but only the phase. It is convenient to introduce the complex order parameter

$$R(t) = |R|e^{i\theta(t)} = \frac{1}{2}(\overline{x} + i\overline{y}) = \frac{1}{N}\sum_j e^{i\varphi_j(t)} \tag{2·2}$$

and to represent the global coupling in terms of R. The absolute value of the order parameter is close to zero for nearly uniform, nonsynchronized distributions, and reaches 1 for strongly synchronized states.

Below we will be interested in linear coupling with and without time delay,[6),7)] and in a nonlinear coupling:[5)]

$$\varepsilon' F(\overline{x}, \overline{y}) = 2\omega_0\varepsilon\overline{y}(t) + 2\omega_0\varepsilon_f\overline{y}(t-T) + \frac{d}{dt}(\overline{x}^2(t-T))(K_x\overline{x}(t) + K_y\overline{y}(t))\ .$$

As a result, the phase equations for the oscillators read

$$\dot{\varphi}_i = \omega_i + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin(\varphi_j(t) - \varphi_i(t)) + \frac{\varepsilon_f}{N}\sum_{j=1}^{N}\sin(\varphi_j(t-T) - \varphi_i(t))$$
$$+\varepsilon_{of}|R|^2(t-T)|R|(t)\sin[2\theta(t-T) - \theta(t) - \varphi_i(t) + \nu] + \xi_i(t), \tag{2·3}$$

where $\varepsilon_{of}e^{i\nu} = 2(K_x + iK_y)$. Here three coupling parameters describe different types of coupling: ε describes collective linear coupling without delay, as in the original Kuramoto model; ε_f describes linear coupling with delay, as has been proposed in 6) and 7); ε_{of} describes nonlinear coupling with delay as has been proposed in 5).

§3. Linear feedback: thermodynamic limit and stability

We start with a consideration of an ensemble of oscillators with linear couplings, i.e. in this and the next sections we consider (2·3) with $\varepsilon_{of} = 0$. In the thermodynamic limit $N \to \infty$ we can introduce a distribution of natural frequencies $g(\omega)$ and rewrite system (2·3) as

$$\dot{\varphi}(\omega) = \omega + \varepsilon\int_{-\infty}^{+\infty} g(\omega')\sin\left(\varphi(\omega', t) - \varphi(\omega, t)\right)\, d\omega'$$
$$+\varepsilon_f\int_{-\infty}^{+\infty} g(\omega')\sin\left(\varphi(\omega', t-T) - \varphi(\omega, t)\right)\, d\omega' + \xi(\omega, t)\ . \tag{3·1}$$

For a statistical description one introduces a distribution density $\rho(\omega, \varphi, t)$ (normalized as $\int_0^{2\pi}\rho(\omega, \varphi, t)\, d\varphi = 1$) that is governed by the Fokker-Planck equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\varphi}(\rho\, v) - D\frac{\partial^2\rho}{\partial\varphi^2} = 0, \tag{3·2}$$

where

$$v(\omega) = \omega + \varepsilon \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \rho(\omega', \theta, t) + \varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \rho(\omega', \theta, t - T) \,. \tag{3·3}$$

The order parameter introduced in (2·2) now takes the form

$$R(t) = \frac{1}{N} \sum_j e^{i\varphi_j(t)} = \int_{-\infty}^{+\infty} d\omega \, g(\omega) \int_0^{2\pi} d\varphi \, \rho(\omega, \varphi, t) \, e^{i\varphi}. \tag{3·4}$$

Here we shortly discuss a linear stability analysis of the absolutely nonsynchronous state $\rho_0 = \frac{1}{2\pi}$. Infinitesimal perturbations ρ_1 of this state are governed by the linearization of Eq. (3·2)

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial \varphi} + v_0 \frac{\partial \rho_1}{\partial \varphi} - D \frac{\partial^2 \rho_1}{\partial \varphi^2} = 0 \tag{3·5}$$

with

$$\frac{\partial v_1}{\partial \varphi} = -\varepsilon \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \cos(\theta - \varphi) \rho_1(\omega', \theta, t) - \varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \cos(\theta - \varphi) \rho_1(\omega', \theta, t - T). \tag{3·6}$$

Substituting $\rho_1 = \sum_k c_k(\omega) e^{ik\varphi + \lambda t}$ $(k \neq 0)$, one finds independent equations for different c_k:

$$(\lambda + ik\omega + Dk^2) c_k(\omega) = \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2} (\delta_{k,1} + \delta_{k,-1}) C_k, \tag{3·7}$$

where $C_k = \int_{-\infty}^{+\infty} g(\omega) c_k(\omega) \, d\omega$. Modes with $|k| \neq 1$ always decay while for $k = 1$ one finds

$$c_1(\omega) = \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2(\lambda + D + i\omega)} C_1. \tag{3·8}$$

Multiplying this equation by $g(\omega)$ and integrating over ω, one finds that the spectrum is formed by the roots of the "spectral function" $\Lambda(\lambda)$

$$\Lambda(\lambda) \equiv 1 - \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2} \int_{-\infty}^{+\infty} \frac{g(\omega) \, d\omega}{D + \lambda + i\omega} = 0. \tag{3·9}$$

Generally, $\Im \left(\int_{-\infty}^{+\infty} g(\omega)(D + i\omega)^{-1} \, d\omega \right) = \int_{-\infty}^{+\infty} \omega g(\omega)(D^2 + \omega^2)^{-1} \, d\omega \neq 0$; therefore real roots of $\Lambda(\lambda)$ (including $\lambda = 0$) are not admitted and only one complex root $\lambda = -i\Omega$ with the corresponding mode $\rho_1 = \alpha(\omega) e^{i(\varphi - \Omega t)} + cc$ determines linear stability. From the linear analysis we thus expect a Hopf bifurcation for the transition to synchrony.

In the degenerated case $\int_{-\infty}^{+\infty} \omega g(\omega)(D^2 + \omega^2)^{-1} \, d\omega = 0$, a relation $\Lambda^*(\lambda) = \Lambda(\lambda^*)$ holds, then real roots are admitted and complex roots appear in pairs (λ, λ^*). We expect that in real applications the degeneracy of the frequency distribution is absent, so we do not consider this situation below.

§4. Weakly nonlinear analysis

In this section we perform a weakly nonlinear analysis of the synchronization transition, considering ε as a bifurcation parameter. We write $\varepsilon = \varepsilon_0 + \kappa^2\varepsilon_2$ where ε_0 is the critical value of ε and κ is a small parameter, and represent the probability distribution $\rho(x,t)$ as $\rho_0 + \kappa\rho_1 + \kappa^2\rho_2 + \kappa^3\rho_3 + \dots$. Assuming $\rho_1 = \alpha_1(\omega, t_2, t_4, ...)e^{i(\varphi - \Omega t_0)} + cc$ (here t_k are "slow times") and substituting this in Eq. (3·2) we obtain in the order κ^2 (there are no secular terms in this order):

$$\frac{\partial\rho_2}{\partial t_0} + \rho_0\frac{\partial v_2}{\partial\varphi} + \frac{\partial}{\partial\varphi}(\rho_1 v_1) + v_0\frac{\partial\rho_2}{\partial\varphi} - D\frac{\partial^2\rho_2}{\partial\varphi^2} = 0, \tag{4·1}$$

where

$$\begin{aligned} v_1 &= \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega')\sin(\theta-\varphi)\alpha_1(\omega')\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)e^{i(\theta-\Omega t_0)} + cc \\ &= i\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)A_1 e^{i(\varphi-\Omega t_0)} + cc, \end{aligned} \tag{4·2}$$

and we have introduced $A_j \equiv \int_{-\infty}^{+\infty}\alpha_j(\omega)g(\omega)\,d\omega$. Note that from (3·8) it follows that

$$\alpha_1(\omega) = \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2(\lambda + D + i\omega)}A_1\ . \tag{4·3}$$

This gives the "driving term" in (4·1):

$$\begin{aligned} \frac{\partial}{\partial\varphi}(\rho_1 v_1) &= \frac{\partial}{\partial\varphi}\left(i\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)\alpha_1(\omega)A_1 e^{i2(\varphi-\Omega t_0)} + cc + ...\right) \\ &= -2\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)\alpha_1(\omega)A_1 e^{i2(\varphi-\Omega t_0)} + cc. \end{aligned}$$

Searching for solution of Eq. (4·1) in the form $\rho_2 = \alpha_2(\omega, t_2, t_4, ...)e^{i2(\varphi-\Omega t_0)} + cc$, we obtain, using (4·3),

$$\alpha_2(\omega) = \frac{2\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)A_1\,\alpha_1(\omega)}{-i2\Omega + i2\omega + 4D} = \frac{\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)^2 A_1^2}{2(D + i(\omega-\Omega))(2D + i(\omega-\Omega))}\ . \tag{4·4}$$

In the order κ^3 of Eq. (3·2), secular terms appear:

$$\frac{\partial\rho_3}{\partial t} + \frac{\partial\rho_1}{\partial t_2} + \rho_0\frac{\partial v_3}{\partial\varphi} + \frac{\partial}{\partial\varphi}(\rho_1 v_2 + \rho_2 v_1) + v_0\frac{\partial\rho_3}{\partial\varphi} - D\frac{\partial^2\rho_3}{\partial\varphi^2} = 0. \tag{4·5}$$

Note that $v_2 = 0$ because $\int_0^{2\pi} e^{i\varphi}\rho_2(\omega,\,\varphi,\,t)\,d\varphi = 0$. Calculation of other secular terms yields

$$\begin{aligned} v_3 &= v_{\rho_3} + \varepsilon_2\int_0^{2\pi} d\theta\int_{-\infty}^{+\infty} d\omega' g(\omega')\sin(\theta-\varphi)\rho_1(\omega',\,\theta,\,t) \\ &\quad + \varepsilon_f\int_0^{2\pi} d\theta\int_{-\infty}^{+\infty} d\omega' g(\omega')\sin(\theta-\varphi)\left((-T)\frac{\partial\alpha_1(\omega',\,t_2,...)}{\partial t_2}e^{i(\theta-\Omega(t-T))} + cc\right) \\ &= v_{\rho_3} + \left(i\pi\left(\varepsilon_2 A_1 - \varepsilon_f e^{i\Omega T}T\frac{\partial A_1}{\partial t_2}\right)e^{i(\varphi-\Omega t_0)} + cc\right), \end{aligned}$$

where $v_{\rho_3} = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \left(\varepsilon_0 \rho_3(\omega', \theta, t) + \varepsilon_f \rho_3(\omega', \theta, t-T)\right)$ and

$$\frac{\partial}{\partial\varphi}(\rho_2 v_1) = \frac{\partial}{\partial\varphi}\left(-i\pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) A_1^* \alpha_2(\omega) e^{i(\varphi - \Omega t_0)} + cc + ...\right)$$
$$= \pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) A_1^* \alpha_2(\omega) e^{i(\varphi - \Omega t_0)} + cc + ... \, .$$

(Here "..." denotes non-secular terms.) Collecting all secular terms, we can write them as

$$\left[\frac{\partial \alpha_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2} A_1 + \frac{\varepsilon_f e^{i\Omega T}}{2} T \frac{\partial A_1}{\partial t_2} + \pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) A_1^* \alpha_2(\omega)\right] e^{i(\varphi - \Omega t_0)} + cc \, . \tag{4·6}$$

Now we have to write out the condition of orthogonality of these terms to the solutions of the conjugated problem, i.e. the condition that the secular part of "driving" vanishes. As soon as the scalar product of τ-time-periodic fields $s(\omega, \varphi, t)$ and $c(\omega, \varphi, t)$ is defined by

$$\langle s, c\rangle \equiv \int_{-\infty}^{+\infty} d\omega\, g(\omega) \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\tau} \frac{dt}{\tau}\, s^*(\omega, \varphi, t)\, c(\omega, \varphi, t), \tag{4·7}$$

the conjugated problem reads

$$\left(-\frac{\partial}{\partial t} - ik\omega + Dk^2\right) c_k(\omega) = \frac{\varepsilon_0 + \varepsilon_f e^{-\lambda T}}{2} (\delta_{k,1} + \delta_{k,-1}) \int_{-\infty}^{+\infty} g(\omega') c_k(\omega')\, d\omega' \tag{4·8}$$

and has a solution

$$\frac{e^{i(\varphi - \Omega t)}}{D - i(\omega - \Omega)}. \tag{4·9}$$

Finally, the orthogonality condition, i.e. the vanishing of the scalar product of (4·9) and (4·6), yields the weakly nonlinear amplitude equation:

$$\int_{-\infty}^{+\infty} \frac{g(\omega)\, d\omega}{D + i(\omega - \Omega)} \left[\frac{\partial \alpha_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2} A_1 + \frac{\varepsilon_f e^{i\Omega T}}{2} T \frac{\partial A_1}{\partial t_2} + \pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) A_1^* \alpha_2(\omega)\right] = 0.$$

Substituting here for α_j and introducing a function $G(z) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{g(\omega)\, d\omega}{\omega - z}$ we obtain

$$\dot{A}_1 = \lambda_2(\varepsilon_0, \Omega) A_1 - P(\varepsilon_0, \Omega) A_1 |A_1|^2, \tag{4·10}$$

where λ_2 is the linear growth rate

$$\lambda_2(\varepsilon, \Omega) = \frac{\varepsilon - \varepsilon_0}{i\pi\left(\varepsilon + \varepsilon_f e^{i\Omega T}\right)^2 G'(\Omega + iD) + \varepsilon_f T e^{i\Omega T}} \tag{4·11}$$

and

$$P(\varepsilon, \Omega) = \frac{\pi^2 \left|\varepsilon + \varepsilon_f e^{i\Omega T}\right|^2 \left(iDG'(\Omega + iD) - G(\Omega + i2D) + G(\Omega + iD)\right)}{D\left(iDG'(\Omega + iD) + \pi^{-1} D\varepsilon_f e^{i\Omega T} T \left(\varepsilon + \varepsilon_f e^{i\Omega T}\right)^{-2}\right)}. \tag{4·12}$$

Equation (4·10) and the expressions (4·11), (4·12) are the main result of our analysis. They give a full description of the effect of the delayed global feedback on the synchronization transition in the ensemble of oscillators. The linear part (4·11) has already been discussed in 6), and the expression (4·12) completes the description of the synchronization transition. Having determined the amplitude A_1 from (4·10), one can find the establishing probability distribution

$$\rho(\omega,\,\varphi,\,t)=\frac{1}{2\pi}\left[1+\frac{\pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)}{D+i(\omega-\Omega)}A_1(t)e^{i(\varphi-\Omega t)}+cc\right.$$
$$\left.+\frac{\pi^2\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}A_1^2(t)e^{i2(\varphi-\Omega t)}+cc+O(A_1^3)\right], \tag{4·13}$$

and the order parameter

$$R(t)=2\pi A_1^* e^{i\Omega t}+O(A_1^3).$$

§5. An example: Lorentz distribution of natural frequencies

The general expressions (4·11) and (4·12) above can be considerably simplified for the Lorentzian distribution

$$g(\omega)=\frac{\gamma}{\pi\left((\omega-\omega_0)^2+\gamma^2\right)}, \tag{5·1}$$

where γ is a characteristic width of the distribution and ω_0 is the mean frequency. In this case

$$G(z)=\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{\omega-z}=\frac{i}{2\pi}\frac{1}{\omega_0-i\gamma-z},$$

where $\Im z$ is assumed to be positive (this holds for $D>0$).

First we obtain explicit expressions for spectrum of the linear problem. Equation (3·9) takes the form

$$1+\frac{i\left(\varepsilon+\varepsilon_f e^{-\beta T+i\Omega T}\right)}{2(\omega_0-\Omega-i(\gamma+D+\beta))}=0\,,$$

where we have substituted $\lambda=\beta-i\Omega$, β and Ω being the real growth rate and the frequency. Separating real and imaginary parts, one can find

$$\Omega=\omega_0-\frac{\varepsilon_f}{2}e^{-\beta T}\sin\Omega T,\qquad \varepsilon=2(\gamma+D+\beta)-\varepsilon_f e^{-\beta T}\cos\Omega T. \tag{5·2}$$

The threshold value ε_0 is determined by $\beta=0$. Substituting the expressions above in (4·11) and (4·12) we obtain

$$\lambda_2(\varepsilon_0,\,\Omega)=\frac{\varepsilon_2}{2+\varepsilon_f T e^{i\Omega T}}, \tag{5·3}$$

$$P(\varepsilon_0,\,\Omega)=\frac{4}{(\varepsilon_0+\varepsilon_f e^{i\Omega T}+2D)(2+\varepsilon_f T e^{i\Omega T})}\,. \tag{5·4}$$

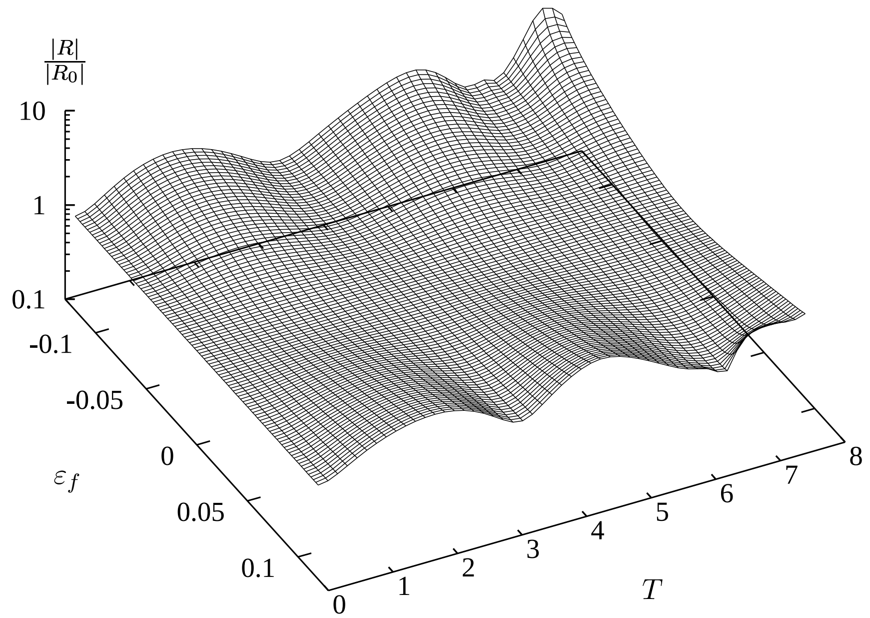

Fig. 1. Effect of delayed feedback on the order parameter for $\omega_0 = 1$, $\gamma = D = 0.01$.

The stationary amplitude A_1 is calculated according to (4·10) $|A_1|^2 = \frac{\Re\lambda_2}{\Re P}$. To demonstrate, how the delayed feedback affects the amplitude, we present in Fig. 1 the ratio $\frac{|R|}{|R_0|}$ where R_0 is the order parameter in the absence of delayed feedback for the same closeness to the transition point ε_2.

§6. Nonlinear delayed feedback

In this section we consider a purely nonlinear delayed feedback in the ensemble of oscillators. We set $\varepsilon_f = 0$ in Eq. (2·3) and write the basic model as

$$\dot{\varphi}_i = \omega_i + \frac{\varepsilon}{N}\sum_{j=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right)$$
$$+\varepsilon_{of}|R|^2(t-T)|R|(t)\sin\left(2\theta(t-T) - \theta(t) - \varphi_i(t) + \nu\right) + \xi_i(t)\,. \qquad (6\cdot1)$$

Similarly to the previous case, in the thermodynamical limit $N \to \infty$ one can write

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\varphi}\left(\rho\, v\right) - D\frac{\partial^2\rho}{\partial\varphi^2} = 0, \qquad (6\cdot2)$$

where

$$v(\omega) = \omega + \varepsilon\int_0^{2\pi} d\varphi' \int_{-\infty}^{+\infty} d\omega' g(\omega')\sin(\varphi' - \varphi)\rho(\omega',\,\varphi',\,t)$$
$$+\varepsilon_{of}|R|^2(t-T)|R|(t)\sin\left(2\theta(t-T) - \theta(t) - \varphi + \nu\right). \qquad (6\cdot3)$$

The linear problem is the same as in the previous case where one sets $\varepsilon_f = 0$. Therefore as soon as $g(\omega_0+\Delta\omega) = g(\omega_0-\Delta\omega)$, critical perturbations either have the frequency ω_0 or are degenerate: they appear in pairs $\omega_0-\Delta\omega$, $\omega_0+\Delta\omega$ (see discussion by Crawford[1)]). We restrict ourselves to non-degenerate case only. Considering nearly critical behavior of small perturbation $\rho_1 = \alpha_1(\omega, t_2, t_4, ...)e^{i(\varphi-\Omega t_0)} + cc$, one can write down from Eq. (6·2) in the order κ^2 (there is no secular terms in this order) Eq. (4·1) with

$$v_1 = i\pi\varepsilon_0 A_1 e^{i(\varphi-\Omega t_0)} + cc. \tag{6·4}$$

Now from Eq. (3·8), $\alpha_1(\omega) = \dfrac{\varepsilon_0\, A_1}{2(D+i(\omega-\Omega))}$. Therefore

$$\frac{\partial}{\partial\varphi}\left(\rho_1 v_1\right) = -\pi\varepsilon_0^2\left[\frac{A_1^2 e^{i2(\varphi-\Omega t_0)}}{D+i(\omega-\Omega)} + cc\right].$$

Searching for ρ_2 in the form $\rho_2 = \alpha_2(\omega, t_2, t_4, ...)e^{i2(\varphi-\Omega t_0)} + cc$ we find

$$\alpha_2(\omega) = \frac{\pi\varepsilon_0^2 A_1^2}{2(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}.$$

In the order κ^3 Eq. (4·5) with $v_2 = 0$ and

$$\begin{aligned} v_3 &= i\pi\left(\varepsilon_0 A_3 + \varepsilon_2 A_1\right) e^{i(\varphi-\Omega t_0)} + cc \\ &\quad + \varepsilon_{of}|R|^2(t-T)|R|(t)\sin\left(2\theta(t-T) - \theta(t) - \varphi + \nu\right) \\ &= i\pi\left(\varepsilon_0 A_3 + \varepsilon_2 A_1\right) e^{i(\varphi-\Omega t_0)} + cc + \varepsilon_{of}\Im\left(R^2(t-T)R^*(t)e^{\nu-\varphi}\right) \end{aligned}$$

is valid. Substituting $R_1 = 2\pi A_1^* e^{i\Omega t}$ we get

$$R_1^2(t-T)R_1^*(t) = 8\pi^3 A_1^{*2} e^{i2\Omega(t-T)} A_1 e^{-i\Omega t} = 8\pi^3|A_1|^2 A_1^* e^{i\Omega(t-2T)}.$$

Therefore

$$v_3 = ... - 8\pi^3\varepsilon_{of}|A_1|^2\Im\left[A_1 e^{i(\varphi-\Omega t-\nu+2\Omega T)}\right]$$

and

$$\rho_0\frac{\partial v_3}{\partial\varphi} = ... - 4\pi^2\varepsilon_{of}|A_1|^2\Re\left[A_1 e^{i(\varphi-\Omega t-\nu+2\Omega T)}\right],$$

where "..." denotes the terms which do not contribute to the secular part of the equation. The term $\frac{\partial}{\partial\varphi}(\rho_2 v_1)$ can be taken from §4, and its contribution to $P(\varepsilon, \Omega)$ is given by the formula (4·12) with $\varepsilon_f = 0$. Summing up these results, one can find that Eq. (4·10) holds with

$$\lambda_2(\varepsilon, \Omega) = \frac{\varepsilon_2}{i\pi\varepsilon^2\, G'(\Omega+iD)}, \tag{6·5}$$

$$P(\varepsilon, \Omega) = \frac{\pi^2\varepsilon^2}{D}\left[1 + \frac{G(\Omega+iD) - G(\Omega+2iD)}{iDG'(\Omega+iD)}\right] + \frac{i4\pi\varepsilon_{of}e^{i(2\Omega T-\nu)}}{\varepsilon G'(\Omega+iD)}. \tag{6·6}$$

The resulting probability density reads

$$\rho(\omega,\,\varphi,\,t)=\frac{1}{2\pi}\left[1+\frac{\pi\varepsilon_0\,A_1(t)}{D+i(\omega-\Omega)}e^{i(\varphi-\Omega t)}+cc\right.$$
$$\left.+\frac{\pi^2\varepsilon_0^2\,A_1^2(t)}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}e^{i2(\varphi-\Omega t)}+cc+O(A_1^3)\right] \quad (6\cdot7)$$

and the order parameter is $R(t)=2\pi A_1^* e^{i\Omega t}+O(A_1^3)$.

As a particular example we consider, like in §2, the Lorentzian distribution of natural frequencies (5·1). The characteristic equation $\Lambda(\lambda)=0$ takes the form

$$\varepsilon-2(\gamma+D)=2(\lambda-i\omega_0) \quad (6\cdot8)$$

and has only one root. The bifurcation of the non-synchronous state is a Hopf one at $\varepsilon_0=2(\gamma+D)$ with the frequency $\Omega=\omega_0$ (see discussion by Crawford[1)]). Setting $\Omega=\omega_0$ in (6·5) and (6·6), we find

$$\lambda_2(\varepsilon_0,\omega_0)=\frac{\varepsilon_2}{2},\qquad P(\varepsilon_0,\omega_0)=\frac{1}{2D+\gamma}-4\pi^2\varepsilon_{of}e^{-i\nu}(\gamma+D). \quad (6\cdot9)$$

The real part of P determines, according to (4·10), the amplitude of the establishing collective mode $|A_1|^2=\lambda_2(\Re P)^{-1}$, with

$$\Re P(\varepsilon_0,\omega_0)=\frac{1}{2D+\gamma}-4\pi^2\varepsilon_{of}(\gamma+D)\cos(\nu)\;.$$

One can see that depending on the value of ν, the amplitude decreases or increases due to additional nonlinear feedback. Moreover, for strong enough feedback $\Re P$ can become negative, what means a subcritical Kuramoto transition. Also, a nonlinear shift of the rotation frequency of R in the counterclockwise direction appears

$$\omega_2=\Im(P)\,|A_1|^2=\frac{\varepsilon_2\Im(P)}{2\,\Re(P)}=\frac{\varepsilon_2}{2}\frac{\tan\nu}{\left[4\pi^2\varepsilon_{of}(2D-\gamma)(D+\gamma)\cos\nu\right]^{-1}-1}. \quad (6\cdot10)$$

§7. Conclusion

In this paper we have developed a weakly nonlinear analysis of the effect of delayed feedback on the Kuramoto transition. We have restricted our attention to the most general case of Hopf bifurcation and have not considered other types of transition that occur under certain symmetries. The analysis is, of course, restricted to a vicinity of the transition point, moreover, the basic phase-coupling model assumes that all types of coupling are weak. A strong coupling case should be studied numerically.

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