Phase Compactons in Chains of Dispersively Coupled Oscillators

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We study the phase dynamics of a chain of autonomous oscillators with a dispersive coupling. In the quasicontinuum limit the basic discrete model reduces to a Korteveg–de Vries-like equation, but with a nonlinear dispersion. The system supports compactons: solitary waves with a compact support and kovatons which are compact formations of glued together kink-antikink pairs that may assume an arbitrary width. These robust objects seem to collide elastically and, together with wave trains, are the building blocks of the dynamics for typical initial conditions. Numerical studies of the complex Ginzburg-Landau and Van der Pol lattices show that the presence of a nondispersive coupling does not affect kovatons, but causes a damping and deceleration or growth and acceleration of compactons.

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Introduction.—Coupled self-sustained oscillators are of fundamental importance in biology, electronics, chemical reactions, optics, acoustics. Quite often oscillators are organized in lattices of elements with nearest neighbors coupling as in laser arrays [1], Josephson junctions [2], or phase locked loops [3], and exhibit a wide range of phenomena from synchronization to space-time chaos. In spite of the particulars of each system, many general features can be described in a framework of coupled phase equations [4,5]. Unlike the amplitude which relaxes to a particular value, the phase of a self-sustained oscillator is free. Thus, weak couplings impact only the phases.

In this Letter, we study chains of limit-cycle oscillators with a dispersive coupling and demonstrate that the building blocks of their spatiotemporal phase dynamics consist of *discrete compactons*—solitary excitations with an almost compact spatial support, which in the quasicontinuum limit become strictly compact. Compactons are basic excitations of Korteveg—de Vries (KdV)-like equations with a nonlinear dispersion [6,7] and other nonlinear dispersive systems [8]. However in the present case the phase space of the N oscillator is an N-dimensional torus and thus the amplitudes are always bounded. This begets a new type of solitary compact pulse with a flattop which we refer to as *the kovaton* (after the Hebrew "kova" for a hat), which attains both the maximal amplitude and speed the system sustains. All kovatons share the same height and speed but, unlike the presented compactons whose width depends on their amplitude, kovatons width is solely determined by the initial state they evolve from.

We start our discussion with the Van der Pol lattice, a realistic paradigm of coupled limit-cycle oscillators,

$$\ddot{x}_n - \mu(1 - x_n^2)\dot{x}_n + x_n = \mu\delta\Delta_d x_n. \tag{1}$$

Here μ is a parameter determining the proximity to the Hopf bifurcation point and the slow time scale, δ describes coupling strength, and $\Delta_d x_n = x_{n+1} + x_{n-1} - 2x_n$ is the discrete Laplacian operator. We define the phase at each site via $\phi = \arctan(\dot{x}/x)$ and present numerically obtained space-time plots of phase differences $v_n = \phi_n - \phi_{n-1}$ in Fig. 1. Note the localized in space, particlelike objects evolving from a localized initial profile; these objects are the *phase compactons*. Remarkably, their amplitude does not remain constant, but will grow and accelerate for $\delta < \delta_c \approx 0.0725$, and decay and decelerate for $\delta > \delta_c$.

Basic Model.—Consider a chain of limit-cycle oscillators. When the coupling is weak, one neglects the changes in the amplitudes and studies only the phase dynamics [4], given via

$$\dot{\phi}_n = \omega + q(\phi_{n-1} - \phi_n) + q(\phi_{n+1} - \phi_n), \tag{2}$$

where q is a 2π-periodic coupling function. Alternatively, for phase differences $v_n = \phi_n - \phi_{n-1}$ we obtain

$$\dot{v}_n = q(-v_n) + q(v_{n+1}) - q(-v_{n-1}) - q(v_n). \tag{3}$$

If q is an odd function, as is typically assumed in the context of synchronization studies [9], the coupling is dissipative. In contradistinction, we shall use even q which induces dispersion (reactive coupling). Coupling functions

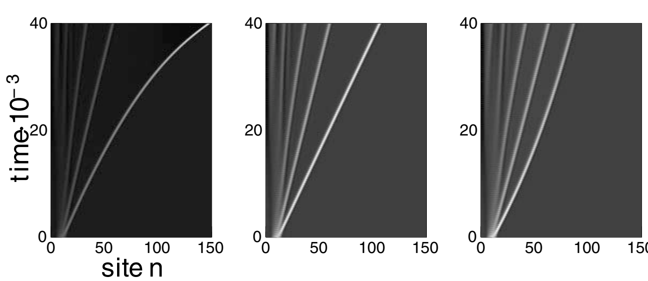

FIG. 1. Compactons evolving from an initial perturbation of the Van der Pol chain (1) for $\mu = 0.2$ and three values of parameter δ (from left to right: 0.0675, 0.0725, 0.08). Black: small values of v_n; white: large values of v_n.

that are neither symmetric nor antisymmetric lead to both dispersive and dissipative interaction; this will be discussed for Van der Pol and complex Ginzburg-Landau (CGL) lattices shortly [10]. To elucidate the main effects, we restrict ourselves to the simplest even periodic function $q(\phi) = \cos\phi$ (the amplitude of q can always be scaled to unity). Thus, our basic model is

$$\dot{v}_n = \cos v_{n+1} - \cos v_{n-1} = \nabla_d \cos v_n, \tag{4}$$

where $\nabla_d u_n \equiv u_{n+1} - u_{n-1}$. Next, we note a few elementary features of (4): (i) Any constant $v_n = V_0$ is a solution. Its perturbations propagate as linear waves provided that $V_0 \neq 0, \pi$. (ii) Symmetries: Eq. (4) is invariant under $v \to -v$, $n \to -n$ and $v \to \pi + v$, $n \to -n$. (iii) Conservation laws: $\sum_n v_n = \text{const}$, $\sum_n (-1)^n v_n = \text{const}$, $\sum_n \sin v_n = \text{const}$. (iv) Equation (4) conserves the phase volume.

The small amplitude limit of Eq. (4),

$$\dot{v}_n = (v_{n-1}^2 - v_{n+1}^2)/2, \tag{5}$$

makes the essentially nonlinear nature of Eq. (4) plainly clear. Though Eq. (5) is of independent interest [7], here it is only used to illuminate the small amplitude limit of (4). In particular, the invariance of (5) under $v_n \to \alpha v_n$, $t \to t/\alpha$, implies a slow evolution of small perturbations.

Compactons.—To unfold the structures supported by (4), we replace n with a continuous variable x and expand the finite difference one order beyond the continuum:

$$\partial_t v = 2\partial_x (1 + \partial_{xx}^2/6) \cos v. \tag{6}$$

For small v, Eq. (6) reduces to a quasicontinuous version of (5): $\partial_t v + \partial_x(1 + \frac{1}{6}\partial_{xx}^2)v^2 = 0$ which is the compactons yielding $K(2,2)$ equation [6].

To find traveling waves (TW) $v(s) = v(x - \lambda t)$, we integrate Eq. (6) twice to obtain

$$Q(v)[0.5(dv/ds)^2 + U(v)] = 0. \tag{7}$$

Here $Q(v) = \sin^2(v)$. The presence of Q stresses the vital role of the singularities at $v = 0$ and π. The potential

$$U(v) = 3[(1 - \cos v)^2 + \lambda(v \cos v - \sin v)]/Q \tag{8}$$

enables one to view (7) as an equation for motion of a particle. The two constants of integration are chosen to assure smoothness at $v = 0$. We now define the turning points of U: $v = 0$ and at $v_m < \pi$, where $U(v_m) = 0$. This gives the amplitude-velocity relation $\lambda = (1 - \cos v_m)^2 \times (\sin v_m - v_m \cos v_m)^{-1}$. Thus, Eq. (7) describes a periodic solution $\tilde{v}(s) = \tilde{v}(s + S)$, with $S = 2\int_0^{v_m}[-2U(v)]^{-1/2}dv$ such that $\tilde{v}(s - S/2) = \tilde{v}(s + S/2) = 0$ and $\tilde{v}(0) = v_m$.

To construct a compact solitary wave, we first note that the highest, third-order operator in Eq. (6) degenerates at $v = 0$, hence also the degeneracy of $Q(v)$ in (7). As a consequence the solution's uniqueness is lost and each period of the solution $\tilde{v}(s)$ starting at $\tilde{v} = 0$ is isolated and does not "communicate" with neighboring periods. This allows one to cut out one period of $\tilde{v}(s)$ at the points of the degeneracy at $v = 0$ and connect it with the trivial $v = 0$ state. This yields a *compacton*

$$v_{\text{cmp}}(s) = \begin{cases} \tilde{v}(s) & \text{if}|s| < S/2, \\ 0 & \text{if}|s| > S/2. \end{cases} \tag{9}$$

Note that near the singularity, since $v \sim y^2 H(y)$, where y is the local coordinate and $H(y)$ is the Heaviside function, v_{yy} has a finite jump. However, since $(v^2)_{yyy} \sim yH(y)$, v satisfies Eq. (6) in a conventional sense. Factor $Q(v)$ has a similar, mollifying, role at $v = 0$ in (7). At small amplitudes the compacton coincides with the compacton solution of the $K(2,2)$ model: $v(s) = \frac{4\lambda}{3}\cos^2(\sqrt{3/8}s)$ for $|s| \leq \sqrt{2/3}\pi$ and zero elsewhere.

The Kovaton.—As the compacton's amplitude increases, one arrives at the amplitude $v = \pi$ where $U(\pi) = 0$ and $Q(v)$ vanishes again. The compacton now reaches its maximal velocity $\lambda_c = \frac{4}{\pi}$ and its highest amplitude is $v_m = \pi$. The potential becomes symmetric: $U^c(v) = U^c(\pi - v)$ and this, critical, solution is now singular *both at the bottom and the top*. It is a kink that connects the degenerated states $v = 0$ and $v = \pi$ [see symmetry property (ii)]. Two such compact kinks (or rather a kink-antikink formation) at an arbitrary, but finite, distance from each other form a new compact entity *the kovaton*. In Figs. 2 and 3 both the compacton and kovaton solutions of the model equation (6) are shown.

The Discrete Model.—To study TWs of Eq. (4) we assume $v_n(t) = v(t - n/\lambda)$ in (4) and integrate once:

$$v(t) = \int_{t-1/\lambda}^{t+1/\lambda}[1 - \cos v(x)]dx. \tag{10}$$

This form can be used both for analytical purposes (to prove the existence of TWs), or as a device to calculate numerically their shape. For large t, assuming that v is

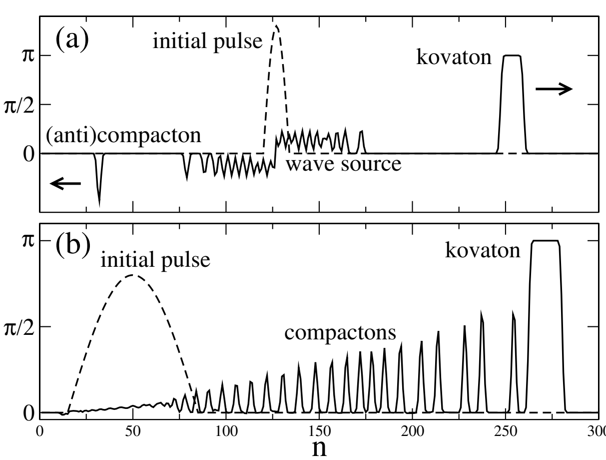

FIG. 2. (a) A localized initial perturbation (dashed curve) evolves into a kovaton which moves to the right, an (anti)compacton that moves to the left, and a source of waves. (b) A wide pulse evolves into a kovaton and a set of compactons.

small and $v(t) \approx e^{-f(t)}$, where $f(t)$ is a rapidly growing function, we estimate the integral à la Watson to obtain $f(t) \approx C \exp(t\lambda \ln 2) - t\lambda \ln 2$, where C is a constant. Thus, the sharp cutoff of the quasicontinuum compactons or kovatons turns into a boundary layer wherein the decay is superexponential (cf. the inset of Fig. 3). This appears to be a generic property of essentially nonlinear discrete interactions. A similar effect was recently shown for a compact breather in a chain of anharmonically coupled mass points [8]. It is indeed notable that though, strictly speaking, the singularities of continuum models disappear in the discrete description, the key properties associated with these singularities—a finite span of the pulse, clean collisions or the uniform height of all kovatons—are preserved by the discrete antecedent. Hence the importance of the quasicontinuum models.

Numerical studies of (4) reveal that a typical localized initial pulse decomposes into three types of objects (see Fig. 2): (i) Discrete compactons of different amplitudes ($|v| < \pi$) moving with corresponding velocities. (ii) Discrete kovatons—hat-shaped configurations of kink-antikink formations connecting 0 and π that move with the critical velocity λ_c and width determined by the initial datum. (iii) Wave sources—standing objects that emit wave trains in opposite directions.

For initial conditions in the form of a localized unimodal pulse (like in all simulations we present here), a small and wide pulse typically produces only compactons (as in Fig. 1), a large and narrow pulse produces all three types of objects [Fig. 2(a)], and a large and wide pulse produces a kovaton followed by compactons [Fig. 2(b)].

In Fig. 3 we compare the numerical solutions of Eq. (4) with the compact structures obtained via (7). The discrete compacton's width is essentially confined to 5–6 sites: outside of 8 sites the field drops to values less than 10^{-10}. We reiterate that the core of the kovaton may assume any width. Its stability is due to the fact that all the kinks propagate with the same critical velocity λ_c.

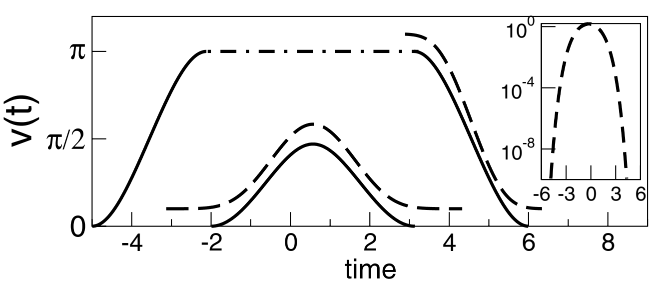

FIG. 3. Solid lines: The compacton and kovaton TW solutions of Eqs. (6) and (7). Dashed lines: the corresponding discrete compacton and kovaton solutions of Eq. (4) (shifted vertically for clarity of presentation, otherwise they can hardly be distinguished from solid ones). The discrete solutions are represented at a particular site as functions of time. Inset: the discrete compacton in linear-logarithmic coordinates.

Clearly, due to symmetry (ii) we also have anticompactons and antikovatons on the base of $v = 0$ having negative amplitude and thus moving to the left, and corresponding compatriots on the $v = \pi$ base.

We now discuss the compacton and the kovaton solutions in terms of the original oscillator lattice. The state $v = 0$ represents a uniform phase of the lattice. A compacton on this base describes a phase step that moves with a constant velocity between two uniform phases. The total phase shift is the sum of values of v over the compacton [in the continuous approximation it is the integral of (9)]. The maximal possible phase shift corresponds to the compacton with the maximal amplitude, which is a kovaton with zero width at the top. A numerical estimate of this shift yields $(\Delta\phi)_c \approx \frac{7}{4}\pi$. The state $v = \pi$ represents a lattice wherein the neighboring oscillators are in antiphase. The kink is then a front that propagates between uniform in-phase and antiphase states. The kovaton describes *a moving domain* of an antiphase lattice, with the total phase shift between the uniform phases on both sides of the domain being $(\Delta\phi)_c + m\pi$. Compactons and anticompactons on the base of the $v = \pi$ state have a similar meaning.

Collisions.—Numerical experiments (Fig. 4) reveal that compactons and kovatons remain intact after a collision without measurable change of amplitude or shape. However, similarly to other compacton carrying systems [6], the collision site is marked by a small amplitude "zero mass" residue which slowly evolves (a consequence of the aforementioned inverse time-amplitude scaling) and may shed new compacton-anticompacton pairs. A finite periodic chain will eventually be completely marked by such sites [10].

CGL and Van der Pol Chains.—Next we consider a simple lattice analog of the complex Ginzburg-Landau equation (CGLL) [11]:

$$\dot{A}_n = DA_n(1 - |A_n|^2) + i\Delta_d A_n. \tag{11}$$

This corresponds to reactive coupling of isochronous oscillators. [Equation (11) may be also relevant for the description of coupled lasing optical waveguides.] For large D the amplitude $|A|$ is close to its limit-cycle value $|A| = 1$ and in the limit $D \to \infty$ yields the phase model (2). To

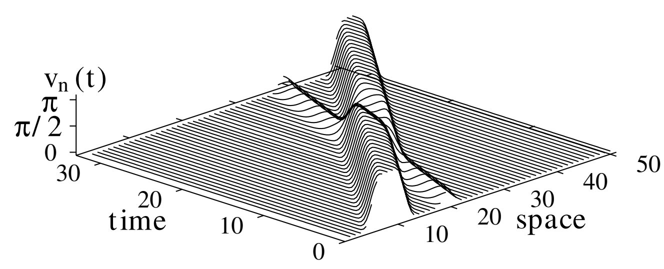

FIG. 4. Collision of a kovaton and a compacton. Note the phase shift of the reemerging pulses which remain intact and the hardly noticeable residue left at the collision site.

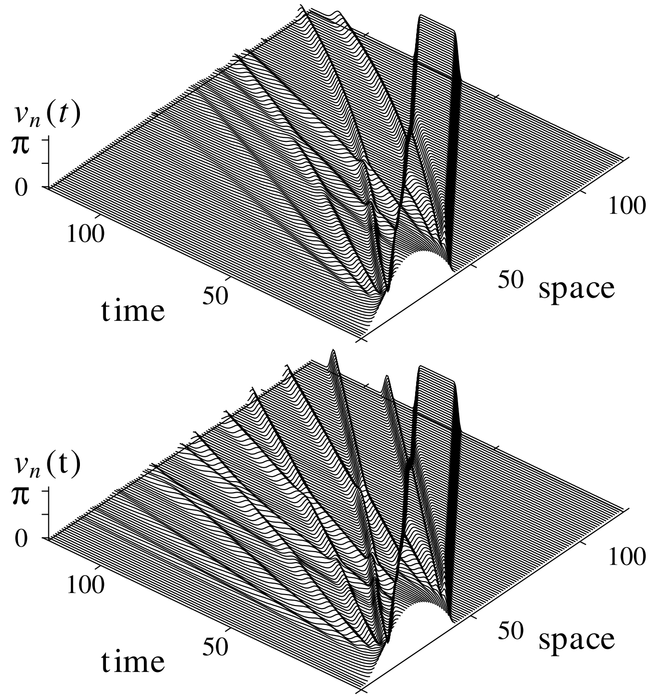

FIG. 5. Compactons that evolve from an initial perturbation in a CGLL (11) decay and decelerate for $D = 30$ (upper panel) but propagate without notable losses for $D = 600$ (bottom panel). In both cases the kovaton appears *unaffected*.

obtain a correction of order D^{-1}, we write $A_n = (1 + r_n)e^{i\phi_n}$, where r_n is of order D^{-1}, and obtain

$$\frac{dv_n}{dt} = \nabla_d \cos v_n - \frac{1}{2D}\Delta_d(\cos v_n \Delta_d \sin v_n). \tag{12}$$

Note that the last term in (12) is dissipative for $v_n < \pi/2$, but antidissipative for $\pi/2 < v_n < \pi$. Thus, large-amplitude compactons decay and decelerate at a slower rate than their smaller siblings. Notably, simulations of (12) reveal that for the kovaton dissipation and antidissipation have a canceling effect and it propagates with a constant speed even for relatively small values of D (see Fig. 5).

Returning to the Van der Pol chain (1), we note that the dynamics shown in Fig. 1 clearly indicates the presence of a passive/active part in addition to a dispersive coupling. This is best seen if one follows the leading compacton: for $\delta = 0.0675$ it grows and accelerates, but for $\delta = 0.08$ it decreases and decelerates. This means that the coupling in Eq. (3) has both even and odd parities, with the latter changing its sign near δ_c. Indeed, the corresponding phase equation now reads [10]

$$\frac{2}{\mu\delta}\frac{dv_n}{dt} = \nabla_d \cos v_n - \frac{3\mu}{4}\Delta_d \sin v_n - \frac{\delta}{2}\Delta_d(\cos v_n \Delta_d \sin v_n). \tag{13}$$

Here, for small v, the term $\sim\mu$ on the right-hand side is active and the term $\sim\delta$ is dissipative, which explains the existence of the aforementioned critical value δ_c.

Summary.—Using the phase approximation, we have shown that chains of coupled limit-cycle oscillators support discrete compactons, and a new type of excitations: kovaton, consisting of a compact kink-antikink pair that connects two trivial $v = 0$ and $v = \pi$ states. They are the main building blocks of the space-time dynamics.

A "conservative" phase dynamics may seem paradoxical in dissipative systems [or ultradissipative, in the CGLL case (11)]. But this is resolved noting that, in a limit cycle, the phase dynamics of even a strongly dissipative system *has a zero Lyapunov exponent and thus effectively is nondissipative*. In a chain this is possible if the coupling between oscillators is reactive [12].

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