Lyapunov exponents in disordered chaotic systems: Avoided crossing and level statistics

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The behavior of the Lyapunov exponents (LEs) of a disordered system consisting of mutually coupled chaotic maps with different parameters is studied. The LEs are demonstrated to exhibit avoided crossing and level repulsion, qualitatively similar to the behavior of energy levels in quantum chaos. Recent results for the coupling dependence of the LEs of two coupled chaotic systems are used to explain the phenomenon and to derive an approximate expression for the distribution functions of LE spacings. The depletion of the level spacing distribution is shown to be exponentially strong at small values. The results are interpreted in terms of the random matrix theory.

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The Lyapunov exponent (LE), which measures the instability of dynamical trajectories, is a standard tool in the studies of chaotic systems [1]. The positiveness of the maximal LE serves as the criterion for chaos; the inverse LE is a characteristic time of mixing and of correlation decay. For an N-dimensional chaotic system, N LEs corresponding to different directions in the phase space can be defined.

There are different methods to calculate LEs numerically if the equations of motion are known; in experiments one can use special techniques of data analysis to estimate at least some largest LEs from the observed data [2]. Several important physical properties can be expressed in terms of LEs, e.g., the Lyapunov dimension [3] and the synchronization threshold [4,5]. Whereas in chaos the largest LE is most important, in studies of disordered lattices (the Anderson localization problem [6–8]) the smallest, in absolute value, LE of the transfer matrix is important; it gives the inverse localization length.

LEs can be considered as some kind of eigenvalues characterizing chaotic motion. Thus, it appears to be natural to perceive an analogy to other eigenproblems in physics, in particular, to eigenfrequencies of linear oscillator systems and to energy eigenvalues of quantum systems. This analogy has been shown to work for spatially extended chaotic systems. There the LEs can in the thermodynamic limit be described with the spectral density [9,10], similar to the usual description of eigenmodes of lattices. In this paper we use the analogy with energy levels to investigate the properties of LEs in disordered chaotic systems. It has to be emphasized, however, that while energy levels and eigenfrequencies are directly observable quantities, LEs are defined in a theoretical concept and can at most be measured in an indirect way.

A typical model here is a lattice or an ensemble of coupled chaotic systems whose parameters are randomly distributed. Such systems, as has been shown in [11,12], can demonstrate rather unusual properties, e.g., disorder-enhanced synchronization. Here we concentrate on the properties of LEs in disordered systems. The main observation is that these properties resemble those of energy levels in disordered quantum systems, but are quantitatively different. In particular, we demonstrate that the LEs exhibit ''avoided crossing'' when drawn in dependence on a parameter. The second result concerns the statistics of ''level spacings'': we demonstrate that the distribution of the differences between the LEs has an exponentially strong depletion at small differences, in contrast to the Wigner (and similar) distributions in the random matrix theory. We give a theoretical explanation for this depletion, based on the properties of LEs of two coupled chaotic systems demonstrating extremely strong ''level repulsion'' [13,14]. In order to demonstrate qualitative universality of the effects of avoided crossing and level repulsion, we consider below different types of systems, Hamiltonian and dissipative ones, and different couplings, global- and nearest-neighbor-type.

A. Numerical evidence for avoided crossing and level repulsion

Our basic model is a system of N coupled standard maps that are, in general, different;

$$I_i(t+1)=I_i(t)+K_i \sin\theta_i(t)+\frac{\varepsilon}{\mathcal{N}\{j\}}\sum_{\{j\}}\sin[\theta_i(t)-\theta_j(t)], \tag{1a}$$

$$\theta_i(t+1)=\theta_i(t)+I_i(t+1), \quad i=1,\ldots,N. \tag{1b}$$

Here $I_i(t)$ and $\theta_i(t)$ are the 2π-periodic state variables at site i and time t, and ε serves as the coupling parameter. The coupling can be global if the sum on the right-hand side is over all elements in the ensemble, in this case $\mathcal{N}\{j\}=N-1$. In the case of local coupling in a one-dimensional periodic lattice, the sum is over nearest neighbors and $\mathcal{N}\{j\}=2$. The parameters K_i of all systems are, in general, different, their random distribution defines disorder in the model. Below we take all parameters K_i in the region of strong chaos, $K_i>7$. The standard map used in Eq. (1) is the basic model of Hamiltonian chaos [15], it describes, in particular, a periodically kicked rotator.

The LEs are calculated with standard methods [7] as the logarithms of the eigenvalues of the limiting matrix

$$V=\lim_{T\to\infty}[P_T^{\dagger}P_T]^{1/2T}, \quad P_T=\prod_{t=1}^{T}J(t), \tag{2}$$

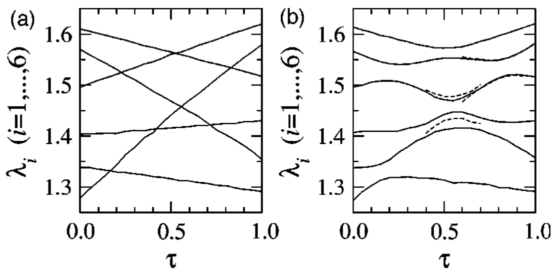

FIG. 1. Lyapunov exponents λ_i ($i=1,\ldots,6$) vs parameter τ (see text) for six standard maps with parameters $K_i(\tau)$. (a) Without coupling crossings of LEs are possible. (b) Crossings are avoided when nearest-neighbor coupling with coupling parameter $\varepsilon=10^{-8}$ is applied. The dashed lines correspond to avoided crossings of only two coupled maps, see text.

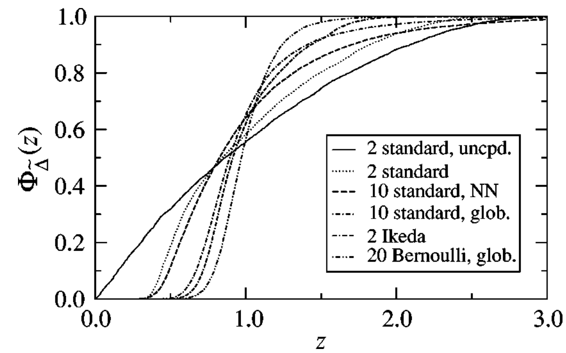

FIG. 2. Numerically estimated cumulative distribution functions $\Phi_{\tilde{\Delta}}(z)$ for the normalized (in such a way that the mean spacing is 1) LE spacings $\tilde{\Delta}$ of different systems (Standard and Bernoulli maps with average coupling parameter $\varepsilon_0=10^{-5}$, Ikeda maps with $\varepsilon_0=10^{-4}$) with different types of coupling (uncpd., uncoupled; NN, nearest neighbor coupling; glob., global coupling).

where $J(t)$ is the Jacobian of the mapping (1). Since the standard map is symplectic, it has (for chaotic trajectories) one positive and one negative LE of the same absolute value that depends on the parameter K.

To demonstrate the avoided crossing of LEs, the maps in Eq. (1) are now considered as depending on a common parameter $\tau\in[0,1]$ as

$$K_i=K_i(\tau)=K_i(0)+\tau[K_i(1)-K_i(0)].$$

The parameters $K_i(0)$ and $K_i(1)$ are random numbers uniformly distributed in the interval $7\leqslant K_i\leqslant 10$. We present in Fig. 1 the results of numerical calculations of the LEs of a particular realization of a system [Eq. (1)] of six nearest-neighbor-coupled standard maps. In Fig. 1(a), the six positive LEs λ_i ($i=1,\ldots,6$) are shown as functions of the common parameter τ for the case $\varepsilon=0$, i.e., without coupling. As can be expected for independent LEs, many crossings are observed. This is no longer the case when a small nearest-neighbor coupling ($\varepsilon=10^{-8}$) is introduced, as can be seen from Fig. 1(b); the crossings are avoided, a behavior that is well known for energy levels of quantum-mechanical systems. Note, however, a quantitative difference: Since the LEs are calculated from the eigenvalues of a product of random matrices, the avoided crossing is already observed for extremely small (in absolute value) off-diagonal elements of the single matrices.

A theoretical explanation for this strong repulsion of LEs will be discussed below, here we want to describe further numerical experiments showing that the picture above is quite universal. A qualitatively similar pattern of avoided crossings has also been obtained for a lattice of standard maps with global coupling. We have observed it also for dissipative systems, e.g., for globally coupled skew Bernoulli maps with parameters $a_i\in(0,1)$ defining the location of the discontinuity. Another dissipative system we studied is the Ikeda map for a complex amplitude E,

$$E(t+1)=a+bE(t)\exp\left(ic-\frac{id}{1+|E(t)|^2}\right),$$

which describes a chaotic regime of light propagation in a ring cavity with a nonlinear element [16]. Coupling such systems can be achieved by overlapping the light fields (see, e.g., the experiments [17]). Below we describe the LEs in coupled Ikeda maps, where the disorder is due to different values for the parameters d_i of the different maps, while the other parameters were kept constant ($a=1$, $b=0.9$, $c=0.4$). The Ikeda map has one positive and one negative LE; we follow only the statistics of positive LEs.

Now we demonstrate that the consequence of the LE repulsion is a particular statistics of LE spacings in disordered systems of type (1). We performed the numerical experiment with different kinds of coupled maps as follows. First, we fixed the system size N and the expectation value ε_0 of the coupling constant. Then, for each randomly chosen set of parameters (we used uniformly distributed parameters $K_i\in[7,10]$ for standard maps, $a_i\in[0.2,0.3]$ for skew Bernoulli maps, and $d_i\in[7.5,8.5]$ for Ikeda maps) and coupling constant ε (exponential distribution with expectation value $\varepsilon_0=10^{-5}$ for standard and skew Bernoulli maps, $\varepsilon_0=10^{-4}$ for Ikeda maps), we determined N LEs, which correspond to $N-1$ spacings $\Delta_i=\lambda_i-\lambda_{i+1}$. These spacings are considered as $N-1$ samples of a random distribution (for the standard and Ikeda maps only the positive LEs are considered). Performing calculations for many sets of parameters K_i (or a_i, or d_i) and ε, we obtain a representative statistics for the LE spacings, see Fig. 2 where the distribution function $\Phi_\Delta(z)=\text{Prob}(\Delta<z)$ is shown.

Examining Fig. 2 we see that the distribution of spacings of coupled maps has a very strong depletion for small z, not only compared to the Poisson distribution $\Phi\sim z$ (which occurs in the absence of coupling), but also compared to the Wigner distribution for the Gaussian orthogonal ensemble of random matrices for which $\Phi\sim z^2$ [18].

To resolve this strong depletion, we present the data in Fig. 3 in scaled coordinates. The scaling is motivated by our theory (see below) and it shows that the distribution function is exponentially small for small spacings: $\Phi_\Delta(z)\sim\exp(-z^{-1})$. Note also that although the distribution functions are qualitatively similar for different systems, they do not collapse on a single curve. This is an indication for nonuniversality of the LE spacing distribution.

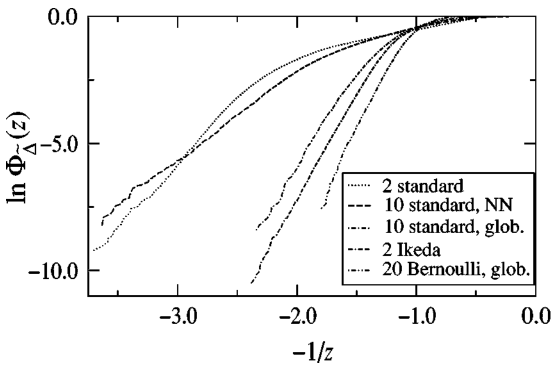

FIG. 3. Cumulative distribution functions of Fig. 2 in scaled coordinates, cf. Eq. (6).

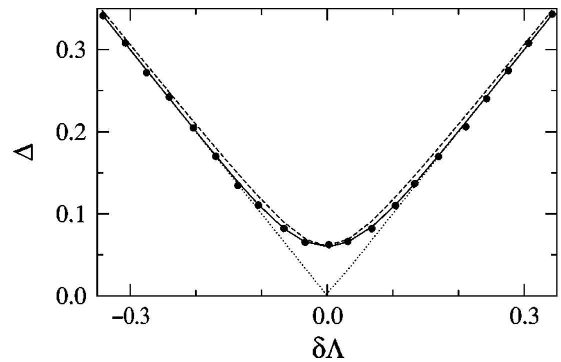

FIG. 4. Dependence of the LE difference Δ on the difference $\delta\Lambda$ between the ''bare'' LEs for two coupled standard maps with coupling parameter $\varepsilon = 10^{-5}$. Comparison of numerical results (circles) with the analytical expression [Eq. (3), solid line, with numerically calculated values for σ^2] and the hyperbolic approximation [Eq. (4), dashed line]. The dotted line depicts the LE difference without coupling, $\Delta = |\delta\Lambda|$.

B. Relation to other random matrix problems

As is clear from Eq. (2), the problem we consider can be formulated as a problem of the random matrix theory (with the usual modeling of chaotic fluctuations with random ones). Namely, we are interested in the eigenvalues of infinite products of random matrices, having both *quenched* (time-independent) disorder and *dynamic* (time-dependent) noise. The quenched randomness comes from the distribution of the parameters in the disordered ensemble, e.g., from the distribution of parameters K_i of the standard maps. The dynamic noise comes from fluctuations due to chaotic evolution (e.g., in the standard map the local Jacobian depends on the chaotic variables I and θ).

The two limiting cases, when our problem can be reduced to standard ones, are clear. In the case when the quenched disorder is absent (or if we consider just one realization of parameters of the interacting chaotic systems), we have a standard problem of the calculation of LEs for a product of random matrices [7]. Another well-known situation appears if the dynamic noise is absent, in this case all the matrices of the product are equal and the problem reduces to calculation of the eigenvalues of this matrix. This problem has been widely discussed, recently mainly in the context of quantum chaos (see, e.g., [18]). For chaotic systems the fluctuations can vanish only in exceptional cases, e.g., for the skew Bernoulli map this happens for the symmetric situation $a_i = 1/2$ only; for the standard map in the chaotic state and for the Ikeda map the fluctuations are always finite. Another limiting case is that of uncoupled systems, here we have a product of diagonal matrices with both quenched and dynamic randomness. The LEs simply follow the statistics of the quenched disorder.

C. Theory

Similar to the case of quantum-mechanical systems (see, e.g., [19]), the essential qualitative and quantitative characteristics of LE repulsion can be acquired from the consideration of two coupled dissipative chaotic systems. We demonstrate this with the following numerical experiment: we calculate the LEs for two coupled maps of Fig. 1, switching off the interaction with other systems. The results for two crossings are shown as dashed lines in Fig. 1. One can see that the behavior of the LEs remains at least qualitatively the same.

Daido [20] has first shown that two coupled identical chaotic systems experience a singular repulsion of the LEs, $\Delta \sim |\ln \varepsilon|^{-1}$, where Δ is the difference between the LEs and ε is the coupling parameter. Using the Langevin approach (i.e., modeling the chaotic fluctuations by a random force with appropriate mean and variance), we have recently derived [13] (cf. [14]) a general expression for Δ that is also valid for nonidentical systems,

$$\Delta(l,\varepsilon,\sigma) = \varepsilon \frac{K_{1-l}(\varepsilon/\sigma^2) + K_{1+l}(\varepsilon/\sigma^2)}{K_l(\varepsilon/\sigma^2)}, \tag{3}$$

where σ^2 is the variance of the finite-time LE, $l = |\delta\Lambda|/2\sigma^2$ is proportional to the difference of the ''bare'' (i.e., without coupling) LEs $\Lambda_{1,2}$ of the interacting systems, and K are the modified Bessel functions [21]. Although Eq. (3) was obtained in the continuous-time Langevin approximation where the fluctuations of the LEs are modeled with Gaussian white noise (thus discarding all temporal correlations), it very well describes the coupled standard maps (Fig. 4) as well as other chaotic systems [13,14]. Because expression (3) is rather inconvenient for further analysis, we use a hyperbolic approximation for it,

$$\Delta^2 \approx (\delta\Lambda)^2 + \left(\frac{2\sigma^2}{\ln(\varepsilon/\sigma^2)} \right)^2. \tag{4}$$

The first term on the right-hand side corresponds to the limit $\delta\Lambda \to \infty$, while the second term is based on an expansion of Eq. (3) for $\delta\Lambda = 0$ and small ε/σ^2 [13,14]. From Fig. 4 one can see that this approximation is rather good.

Using Eq. (4) we can show that in a disordered system the probability to observe tiny values of Δ is exponentially small. It is clear that only small values of $\delta\Lambda$ and ε can give small spacings Δ. If we assume that $\delta\Lambda$ and ε are independent random numbers with constant densities near zero, then the distribution function $\Phi_\Delta(z)=\mathrm{Prob}(\Delta<z)$ can be approximated by the integral over the area $A(z)=\{(\delta\Lambda,\varepsilon): (\delta\Lambda)^2+[2\sigma^2/\ln(\varepsilon/\sigma^2)]^2<z^2\}$ leading to

$$\Phi_\Delta(z)\sim\int\int_{A(z)} d(\delta\Lambda)d\varepsilon$$
$$=2\sigma^2\int_{-z}^{z}\exp\left(-\frac{2\sigma^2}{\sqrt{z^2-(\delta\Lambda)^2}}\right)d(\delta\Lambda). \tag{5}$$

Estimating this integral for $2\sigma^2 \gg z$ gives the exponential depletion at small spacings

$$\Phi_\Delta(z)\sim z^{3/2}\exp\left(-\frac{2\sigma^2}{z}\right). \tag{6}$$

The numerically calculated cumulative distribution functions are in conformity with this result, as can be seen from Fig. 3.

The theoretical analysis above is, strictly speaking, restricted to the case of two interacting chaotic systems. Nevertheless, we expect that it works at least qualitatively for large ensembles as well, because we have seen that the LE repulsion is a ''local'' event, where only the two chaotic subsystems whose LEs are close to each other are involved (cf. Fig. 1).

Concluding, we have characterized numerically and theoretically the statistics of the Lyapunov exponents in disordered chaotic systems. Its main feature is the exponential depletion of the distribution function at small spacings between the exponents. This follows directly from the effect of coupling sensitivity of chaos, according to which the repulsion between the LEs is extremely strong. This repulsion manifests itself also in the avoided crossing of LEs, considered as dependent on a parameter. We have demonstrated that the effects of level repulsion and avoided crossing are observed for chaotic systems of different natures, Hamiltonian and dissipative ones. Also the coupling can be of different forms, in particular, qualitatively similar patterns of avoided LE crossings and of the LE spacing distribution function are observed for global and nearest-neighbor couplings in a lattice.

Our framework of consideration was motivated by the analogy to the problem of level statistics in quantum chaos and complex quantum systems [22,19,18]. Qualitatively, the behavior of LEs is quite similar to that of energy levels in quantum chaos. The main difference is that for disordered chaotic dynamical systems we have two sources of randomness, one quenched due to the disorder and one dynamic due to the chaotic fluctuations. Thus, in contrast to the problem of the distribution of eigenvalues of random matrices, we have the problem of the distribution of eigenvalues of the product of random matrices. There are two limiting cases when these two problems are equivalent. One is the case without coupling, where the LEs remain independent random numbers and obey the Poissonian distribution. Another is the case of vanishing fluctuations of the local LEs (no dynamic randomness), here we have one random matrix whose eigenvalues give the LEs.

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