From: *Handbook of Chaos Control*, ed. H. G. Schuster, Wiley-VCH, Weinheim, 1999

12 Phase Synchronization of Regular and Chaotic oscillators

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12.1 Introduction

Synchronization, a basic nonlinear phenomenon, discovered at the beginning of the modern age of science by Huygens [1], is widely encountered in various fields of science, often observed in living nature [2] and finds a lot of engineering applications [3, 4]. In the classical sense, synchronization means adjustment of frequencies of self-sustained oscillators due to a weak interaction. The phase of oscillations may be locked by periodic external force; another situation is the locking of the phases of two interacting oscillators. One can also speak on “frequency entrainment”. Synchronization of periodic systems is pretty well understood [3, 5, 6], effects of noise have been also studied [7]. In the context of interacting *chaotic* oscillators, several effects are usually referred to as “synchronization”. Due to a strong interaction of two (or a large number) of identical chaotic systems, their states can coincide, while the dynamics in time remains chaotic [8, 9]. This effect is called “complete synchronization” of chaotic oscillators. It can be generalized to the case of non-identical systems [9, 10, 11], or that of the interacting subsystems [12, 13, 14]. Another well-studied effect is the “chaos–destroying” synchronization, when a periodic external force acting on a chaotic system destroys chaos and a periodic regime appears [15], or, in the case of an irregular forcing, the driven system follows the behavior of the force [16]. This effect occurs for a relatively strong forcing as well. A characteristic feature of these phenomena is the existence of a threshold coupling value depending on the Lyapunov exponents of individual systems [8, 9, 17, 18].

In this article we concentrate on the recently described effect of *phase synchronization* of chaotic systems, which generalizes the classical notion of phase locking. Indeed, for periodic oscillators only the relation between phases is important, while no restriction on the amplitudes is imposed. Thus, we define phase synchronization of chaotic system as appearance of a certain relation between the phases of interacting systems or between the phase of a system and that of an external force, while the amplitudes can remain chaotic and are, in general, non-correlated. The phenomenon of phase synchronization has been theoretically studied in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. It has been observed in experiments with electronic circuits [30] and lasers [31] and has been detected in physiological systems [28, 32, 33].

We start with reviewing the classical results on synchronization of periodic self-sustained oscillators in sect. 12.2. We use the description based on a circle map and on a rotation number to characterize phase locking and synchronization. The very notion of phase and amplitude of chaotic systems is discussed in Section 12.3. We demonstrate this taking famouse Rössler and Lorenz models as examples. We show also that the dynamics of the phase in chaotic systems is silimar to that in noisy periodic ones. The next section 12.4 is devoted to effects of phase synchronization by periodic external force. We follow both a statistical approach, based on the properties of the invariant distribution in the phase space, and a topological method, where phase locking of individual periodic orbits embedded in chaos is studied. Different aspects of synchronization phenomena in coupled chaotic systems are described in sect. 12.5. Here we give an interpretation of the synchronization transition in terms of the Lyapunov spectrum of chaotic oscillations. We discuss also large systems, such as lattices and globally coupled populations of chaotic oscillators. These theoretical ideas are applied in sect. 12.8 to the data analysys problem. We discuss a possibility to detect phase synchronization in the observed bivariate data, and describe some recent achievments.

12.2 Synchronization of periodic oscillations

In this section we remind basic facts on the synchronization of periodic oscillations (see, e.g.,[34]). Stable periodic oscillations are represented by a stable limit cycle in the phase space, and the dynamics $\phi(t)$ of a phase point on this cycle can be described by

$$\frac{d\phi}{dt} = \omega_0, \tag{12.1}$$

where $\omega_0 = 2\pi/T_0$, and T_0 is the period of the oscillation. It is important that starting from any monotonically growing variable θ on the limit cycle (so that at one rotation θ increases by Θ), one can introduce the phase satisfying Eq. (12.1). Indeed, an arbitrary θ obeys $\dot{\theta} = \gamma(\theta)$ with a periodic "instantaneous frequency" $\gamma(\theta + \Theta) = \gamma(\theta)$. The change of variables $\phi = \omega_0 \int_0^{\theta} [\gamma(\theta)]^{-1} d\theta$ gives the correct phase, with the frequency ω_0 being defined from the condition $2\pi = \omega_0 \int_0^{\Theta} [\nu(\theta)]^{-1} d\theta$. A similar approach leads to correct angle-action variables in Hamiltonian mechanics. We have performed this simple consideration to underline the fact that the notions of the phase and of the phase synchronization are universally applicable to any self-sustained periodic behavior independently on the form of the limit cycle.

From (12.1) it is evident that the phase corresponds to the zero Lyapunov exponent, while negative exponents correspond to the amplitude variables. Note that we do not consider the equations for the amplitudes, as they are not universal.

When a small external periodic force with frequency ν is acting on this periodic oscillator, the amplitude is relatively robust, so that in the first approximation one can neglect variations of the amplitude to obtain for the phase of the oscillator ϕ and the phase of the external force ψ the equations

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi, \psi) \ , \qquad \frac{d\psi}{dt} = \nu \ , \tag{12.2}$$

where $G(\cdot, \cdot)$ is 2π-periodic in both arguments and ε measures the strength of the forcing. For a general method of derivation of Eq. (12.2) see [35]. The system (12.2) describes a motion on a 2-dimensional torus that appears from the limit cycle under periodic perturbation (see Fig. 12.1a,b). If we pick up the phase of oscillations ϕ stroboscopically at times $t_n = n\frac{2\pi}{\nu}$, we get a circle map

$$\phi_{n+1} = \phi_n + \varepsilon g(\phi_n) \tag{12.3}$$

where the 2π-periodic function $g(\phi)$ is defined via the solutions of the system (12.2). According to the theory of circle maps (cf. [34]), the dynamics can be characterized by the winding (rotation) number

$$\rho = \lim_{n\to\infty} \frac{\phi_n - \phi_0}{2\pi n}$$

which is independent on the initial point ϕ_0 and can take rational and irrational values. If it is irrational, then the motion is quasiperiodic and the trajectories are dense on the circle. Otherwise, if $\rho = p/q$, there exists a stable orbit with period q such that $\phi_q = \phi_0 + 2\pi p$. The latter regime is called *phase locking* or *synchronization*. In terms of the continuous-time system (12.2), the winding number is the ratio between the mean derivative of the phase ϕ and the forcing frequency ν

$$\omega =< \frac{d\phi}{dt} >= \rho\nu \ . \tag{12.4}$$

For ρ irrational and rational one has, respectively, a quasiperiodic dense orbit and a resonant stable periodic orbit on the torus (Fig. 12.1a,b).

The main synchronization region where $\omega = \nu$ corresponds to the winding number 1 (or, equivalently, 0 if we apply mod 2π operation to the phase; for frequencies this means that we consider the difference $\omega - \nu$), other synchronization regions are usually rather difficult to observe. A typical picture of synchronization regions, called also “Arnold tongues”, for the circle map (12.3) is shown in Fig. 12.1c.

Several remarks are in order.

1) The concept of phase synchronization can be applied only to *autonomous continuous-time* systems. Indeed, if the system is discrete (i.e. a mapping), its period is an integer, and this integer cannot be adjusted to some other integer in a continuous way. The same is true for forced continuous-time oscillations (e.g. for the forced Duffing oscillator): here the frequency of oscillations is intrinsically coupled to that of the forcing and cannot be adjusted to some other frequency. We can formulate this also as follows: in discrete or forced systems there is no zero Lyapunov exponent, so there is no corresponding marginally stable variable (the phase) that can be governed by small external perturbations.

2) The synchronization condition (12.4) does not mean that the difference between the phase ϕ of an oscillator and that of the external force ψ (or between phases of two oscillators) must be a constant, as is sometimes assumed (see, e.g. [36]). Indeed, (12.2)

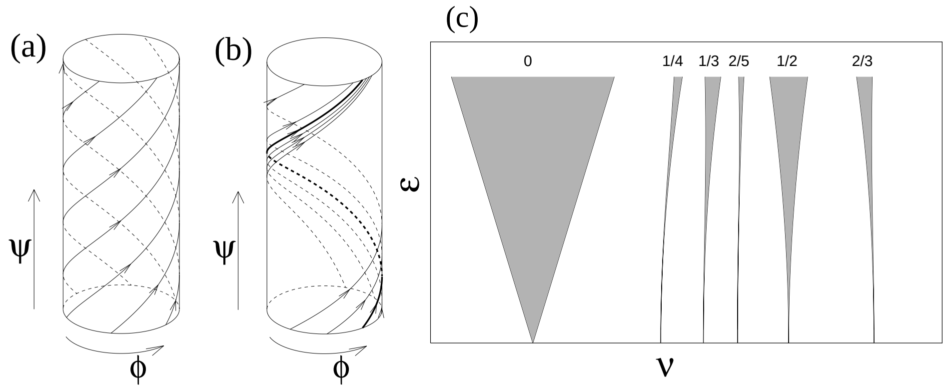

Figure 12.1: Quasiperiodic (a) and periodic flow (b) on the torus; a stable periodic orbit is shown by the bold line. (c): The typical picture of Arnold tongues (with winding numbers atop) for the circle map.

implies, that to enable this, the function G should depend not on separate phases but only on their difference: $G(\phi,\psi) = G(\phi - \psi)$. One can expect that this degeneracy occurs if the form of the oscillations coincides with the form of the external force, e.g. if quasiharmonic oscillations are driven by a sinusoidal force. In general, we can only expect that the deviations of the phase are bounded:

$$|q\phi(t) - p\psi(t)| < \text{const} \ . \tag{12.5}$$

3) The winding number is a continuous function of system parameters; typically it looks like a devil's staircase. Take the main phase-locking region. Continuity means that near the de-synchronization transition the mean oscillation frequency is close to the external one. As the external frequency ν is varied, the de-synchronization transition appears as saddle-node bifurcation, where a stable p/q - periodic orbit collides with the corresponding unstable one, and both disappear. Near this bifurcation point, similarly to the type-I intermittency [37], a trajectory of the system spends a large time in the vicinity of the just disappeared periodic orbits; in the course of time evolution the long epochs when the phases are locked according to (12.5), are interrupted with relatively short time intervals where a phase slip occurs.

4) In the presence of external noise $\xi(t)$ one can consider instead of (12.2) the Langevin equation

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi,\psi) + \xi(t) \ , \qquad \frac{d\psi}{dt} = \nu \ . \tag{12.6}$$

Equivalently, one can model the effect of noise by adding to the mapping (12.3) the noisy term η:

$$\phi_{n+1} = \phi_n + \varepsilon g(\phi_n) + \eta_n \tag{12.7}$$

If the noise is small, the frequencies can be nearly locked, i.e. the averaged relation (12.4) is fulfilled. Large noise can cause phase slips, so that the phase performs a random–walk–

like motion. In the case of unbounded (e.g. Gaussian) noise the mean phase drift is generally non-zero and, strictly speaking, the synchronization region vanishes. Nevertheless, the largest phase-locking intervals survive as regions of nearly constant mean frequency ω. For detailed description of a simple model of synchronization in the presence of noise see [7].

12.3 Phase of a chaotic oscillator

12.3.1 Definition of the phase

The first problem in extending the basic notions from periodic to chaotic oscillations is to properly define a phase. There seems to be no unambiguous and general definition of phase applicable to an arbitrary chaotic process. Roughly speaking, we want to define phase as a variable which is related to the zero Lyapunov exponent of a continuous-time dynamical system with chaotic behavior. Moreover, we want this phase to correspond to the phase of periodic oscillations satisfying (12.1).

To be not too abstract, we illustrate a general approach below on the well-known Rössler system. A projection of the phase portrait of this autonomous 3-dimensional system of ODEs (see eqs. (12.14) below) is shown in Fig. 12.2.

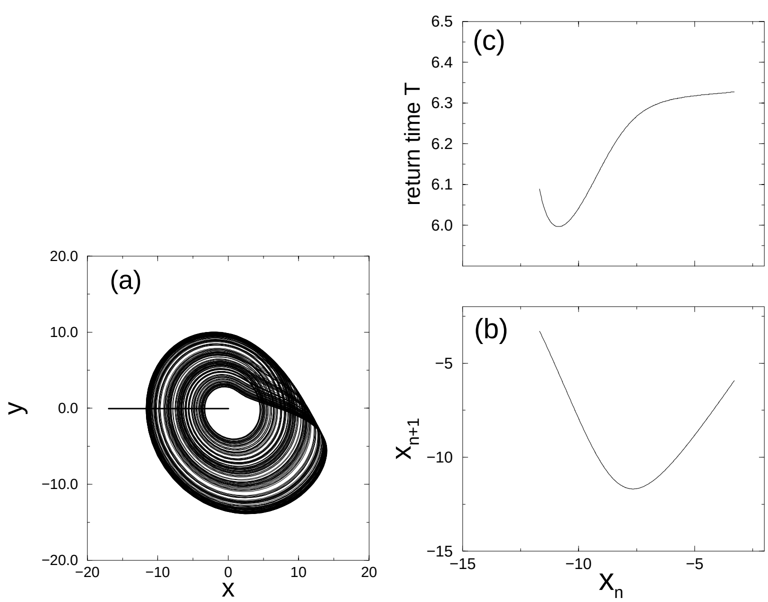

Figure 12.2: Projection of the phase potrait of the Rössler system (a). The horizontal line shows the Poincaré section that is used for computation of the amplitude mapping (b) and dependence of the return time (rotation period) on the amplitude (c).

Suppose we can define a Poincaré map for our autonomous continuous-time system. Then, for each piece of a trajectory between two cross-sections with the Poincaré surface we define the phase just proportional to time, so that the phase increment is 2π at each rotation:

$$\phi(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \qquad t_n \leq t < t_{n+1}. \tag{12.8}$$

Here t_n is the time of the n-th crossing of the secant surface. Note that for periodic oscillations corresponding to a fixed point of the Poincaré map, this definition gives the correct phase satisfying Eq. (12.1). For periodic orbits having many rotations (i.e. corresponding to periodic points of the map) we get a piecewise-linear function of time, moreover, the phase grows by a multiple of 2π during the period. The second property is in fact useful, as it represents the organization of periodic orbits inside the chaos in a proper way. The first property demonstrates that the phase of a chaotic system cannot be defined as unambiguously as for periodic oscillations. In particular, the phase crucially depends on the choice of the Poincaré surface.

Nevertheless, defined in this way, the phase has a physically important property: its perturbations neither grow nor decay in time, so it does correspond to the direction with the zero Lyapunov exponent in the phase space. We note also, that this definition of the phase directly corresponds to the special flow construction which is used in the ergodic theory to describe autonomous continuous-time systems [38].

For the Rössler system Fig. 12.2(a) a proper choice of the Poincaré surface may be the halfplane $y = 0,\ x < 0$. For the amplitude mapping $x_n \to x_{n+1}$ we get a unimodal map Fig. 12.2(b) (the map is essentially one-dimensional, because the coordinate z for the Rössler attractor is nearly constant on the chosen Poincaré surface). In this and in some other cases the phase portrait looks like rotations around a point that can be taken as the origin, so we can also introduce the phase as the angle between the projection of the phase point on the plane and a given direction on the plane (see also [22, 39]):

$$\phi_P = \arctan(y/x)\ . \tag{12.9}$$

Note that although the two phases ϕ and ϕ_P do not coincide microscopically, i.e on a time scale less than the average period of oscillation, they have equal average growth rates. In other words, the mean frequency defined as the average of $d\phi_P/dt$ over large period of time coincides with a straightforward definition of the mean frequency via the average number of crossings of the Poincaré surface per unit time.

12.3.2 Dynamics of the phase of chaotic oscillations

In contrast to the dynamics of the phase of periodic oscillations, the growth of the phase in the chaotic case cannot generally be expected to be uniform. Instead, the instantaneous frequency depends in general on the amplitude. Let us hold to the phase definition based on the Poincaré map, so one can represent the dynamics as (cf. [20])

$$A_{n+1} = M(A_n)\ , \tag{12.10}$$

$$\frac{d\phi}{dt} = \omega(A_n) \equiv \omega_0 + F(A_n)\ . \tag{12.11}$$

As the amplitude A we take the set of coordinates for the point on the secant surface; it does not change during the growth of the phase from 0 to 2π and can be considered as a discrete variable; the transformation M defines the Poincaré map. The phase evolves according to (12.11), where the "instantaneous" frequency $\omega = 2\pi/(t_{n+1} - t_n)$ depends in general on the amplitude. Assuming the chaotic behavior of the amplitudes, we can consider the term $\omega(A_n)$ as a sum of the averaged frequency ω_0 and of some effective noise $F(A)$; in exceptional cases $F(A)$ may vanish. For the Rössler attractor the "period" of the rotations (i.e. the function $2\pi/\omega(A_n)$) is shown in Fig. 12.2(c). This period is not constant, so the function $F(A)$ does not vanish, but the variations of the period are relatively small.

Hence, the Eq. (12.11) is similar to the equation describing the evolution of phase of periodic oscillator in the presence of external noise. Thus, the dynamics of the phase is generally diffusive: for large t one expects

$$< (\phi(t) - \phi(0) - \omega_0 t)^2 > \propto D_p t \ ,$$

where the diffusion constant D_p determines the phase coherence of the chaotic oscillations. Roughly speaking, the diffusion constant is inversely proportional to the width of the spectral peak calculated for the chaotic observable [40].

Generalizing Eq. (12.11) in the spirit of the theory of periodic oscillations to the case of periodic external force, we can write for the phase

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(\phi, \psi) + F(A_n) \ , \qquad \frac{d\psi}{dt} = \nu \ . \tag{12.12}$$

Here we assume that the force is small (of order of ε) so that it affects only the phase, and the amplitude obeys therefore the unperturbed mapping M. This equation is similar to Eq. (12.6), with the amplitude-depending part of the instantaneous frequency playing the role of noise. Thus, we expect that in general the synchronization phenomena for periodically forced chaotic system are similar to those in noisy driven periodic oscillations. One should be aware, however, that the "noisy" term $F(A)$ can be hardly explicitly calculated, and for sure cannot be considered as a Gaussian δ-correlated noise as is commonly assumed in the statistical approaches [7, 41].

12.4 Phase synchronization by external force

12.4.1 Synchronization region

We describe here the effect of phase synchronization of chaotic oscillations by periodic external force, taking as examples two prototypic models of nonlinear dynamics: the Lorenz

$$\begin{aligned} \dot{x} &= 10(y - x), \\ \dot{y} &= 28x - y - xz, \\ \dot{z} &= -8/3 \cdot z + xy + E\cos\nu t. \end{aligned} \tag{12.13}$$

and the Rössler

$$\begin{aligned} \dot{x} &= -y - z + E\cos\nu t \ , \\ \dot{y} &= x + 0.15y \ , \\ \dot{z} &= 0.4 + z(x - 8.5) \ . \end{aligned} \tag{12.14}$$

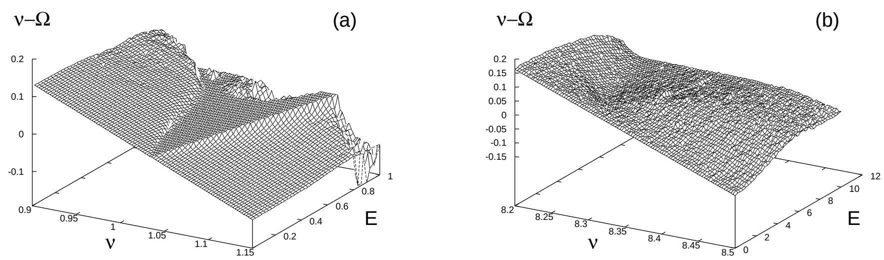

Figure 12.3: The phase synchronization regions for the Rössler (a) and the Lorenz (b) systems.

oscillators. In the absence of forcing, both are 3-dimensional dissipative systems which admit a straightforward construction of the Poincaré maps. Moreover, we can simply use the phase definition (12.9), taking the original variables (x, y) for the Rössler system and the variables $(\sqrt{x^2 + y^2} - u_0, z - z_0)$ for the Lorenz system (where $u_0 = 12\sqrt{2}$ and $z_0 = 27$ are the coordinates of the equilibrium point, the "center of rotation"). The mean rotation frequency can be thus calculated as

$$\Omega = \lim_{t\to\infty} 2\pi \frac{N_t}{t} \tag{12.15}$$

where N_t is the number of crossings of the Poincaré section during observation time t. This method can be straightforwardly applied to the observed time series, in the simplest case one can, e.g., take for N_t the number of maxima (of $x(t)$ for the Rössler system and of $z(t)$ for the Lorenz one).

Dependence of the obtained in this way frequency Ω on the amplitude and frequency of the external force is shown in Fig. 12.3. Synchronization here corresponds to the plateau $\Omega = \nu$. One can see that the synchronization properties of these two systems differ essentially. For the Rössler system there exists a well-expressed region where the systems are perfectly locked. Moreover, there seems to be no amplitude threshold of synchronization (cf. Fig. 12.1c, where the phase-locking regions start at $\varepsilon = 0$). It appears that the phase locking properties of the Rössler system are practically the same as for a periodic oscillator. On the contrary, for the Lorenz system we observe the frequency locking only as a tendency seen at relatively large forcing amplitudes, as this should be expected for oscillators subject to a rather strong noise. In this respect, the difference between Rössler and Lorenz systems can be described in terms of phase diffusion properties (see Sect. 12.3.2). Indeed, the phase diffusion coefficient for autonomous Rössler system is extremely small $D_p < 10^{-4}$, whereas for the Lorenz system it is several oder of magnitude larger, $D_p \approx 0.2$ [24]. This difference in the coherence of the phase of autonomous oscillations implies different response to periodic forcing.

In the following sections we discuss the phase synchronization of chaotic oscillations from the statistical and the topological viewpoints.

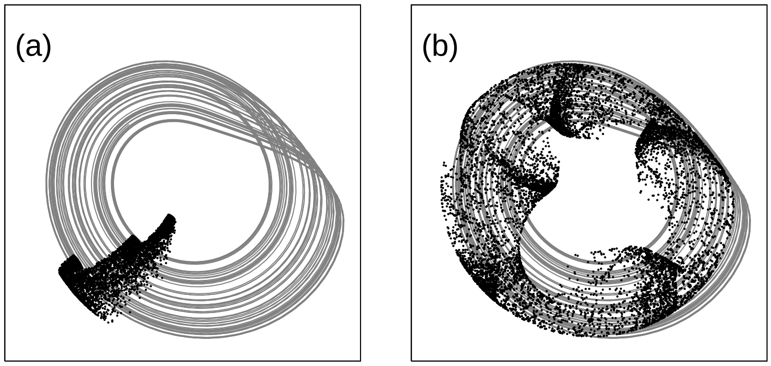

Figure12.4: Distribution inside (a) and outside (b) the synchronization region for the Rössler system, shown with black dots. The autonomous Rössler attractor is shown with gray.

12.4.2 Statistical approach

We define the phase of an autonomous chaotic system as a variable that corresponds to invariance with respect to time shifts. Therefore, the invariant probability distribution as a function of the phase is nearly uniform. This follows from the ergodicity of the system: the probability is proportional to the time a trajectory is spending in a region of the phase space, and according to the definition (12.8) the phase motion is (piecewise) uniform. With external forcing, the invariant measure depends explicitly on time. In the synchronization region we expect that the phase of oscillations nearly follows the phase of the force, while without synchronization there is no definite relation between them. Let us observe the oscillator stroboscopically, at the moments corresponding to some phase ψ_0 of the external force. In the synchronous state the probability distribution of the oscillator phase will be localized near some preferable value (which of course depends on the choice of ψ_0). In the non-synchronous state the phase is spread along the attractor. We illustrate this behavior of the probability density in Fig. 12.4. One can say that synchronization means localization of the probability density near some preferable time-periodic state. In other words, this means appearance of the long-range correlation in time and of the significant discrete component in the power spectrum of oscillations.

Let us consider now the ensemble interpretation of the probability. Suppose we take a large ensemble of identical copies of the chaotic oscillator which differ only by their initial states, and let them evolve under the same periodic forcing. After the transient, the projections of the phase state of each oscillator onto the plane x, y form the cloud that exactly corresponds to the probability density. Let us now consider the ensemble average of some observable. Without synchronization the cloud is spread over the projection of the attractor (Fig. 12.4b), and the average is small: no significant average field is observed. In the synchronous state the probability is localized (Fig. 12.4a), so the average is close to some middle point of the cloud; this point rotates with the frequency ν and one observes large regular oscillations of the average field. Hence, the synchronization can be easily indicated through the appearance of a large (macroscopic) mean field in the ensemble.

Physically, this effect is rather clear: unforced chaotic oscillators are not coherent due to internal chaos, thus the summation of their fields yields a small quantity. Being synchronized, the oscillators become coherent with the external force and thereby with each other, so the coherent summation of their fields produces a large mean field.

An important consequence of the statistical approach described above is that the phase synchronization can be characterized without explicit computation of the phase and/or the mean frequency: it can be indicated implicitly by the appearance of a macroscopic mean field in the ensemble of oscillators, or by the appearance of the large discrete component in the spectrum. Although there may be other mechanisms leading to the appearance of macroscopic order, the phase synchronization appears to be one of the most common ones.

12.4.3 Interpretation through embedded periodic orbits

In order to understand structural metamorphoses of attracting chaotic sets under the action of the synchronizing force, it is convenient to look at the properties of individual periodic orbits embedded into the strange attractors. Unstable periodic orbits are known to build a kind of "skeletons" for chaotic sets [34]; in particular, each of the systems (12.13) and (12.14) in the absence of forcing possesses infinite number of periodic solutions with two-dimensional unstable manifolds. Let us pick up one of these solutions and consider the dynamics on its two-dimensional global stable manifold. From this point of view, there is no difference from familiar problem of the synchronization of stable periodic oscillations by external driving force (see Sect. 12.2 above): the winding number ρ can be introduced, and in the parameter space one should observe synchronization inside the Arnold tongues (locking regions) which correspond to rational values of ρ (cf. Fig. 12.1). Like in the situation described in Sect. 12.2 above, an invariant torus evolves from the periodic orbit of the autonomous chaotic system. Trajectories wind around this torus; inside the Arnold tongues there are two closed orbits on its surface: the attracting one which will call below "phase-stable", and the repelling one, called "phase-unstable". On the border of the locking region these two orbits coalesce and disappear via the tangent bifurcation. Outside the tongues the motion corresponding to this particular periodic orbit is not synchronized and the trajectories are dense on the torus.

Since in the entire phase space of the autonomous system the considered periodic solution is unstable, the torus in the weakly driven system is also unstable. On the plane of the parameters E and ν the tip of the main Arnold tongue lies in the point $E = 0, \nu = \omega_i$ where ω_i is the individual mean frequency of the considered autonomous orbit.This frequency differs from the formally defined frequency of the periodic solution $2\pi/T_i$ where T_i is a period of the orbit: we take here into account also the number of round trips n_i of the orbit and write $\omega_i = 2\pi n_i/T_i$). Naturally, the values of ω_i differ for different periodic orbits; however, in many autonomous dissipative systems (like in Eq. (12.14)) chaos manifests itself in the form of nearly isochronous rotations, and the frequencies ω_i are very close to each other. Respectively, the Arnold tongues overlap (Fig. 12.5), and one can find the parameter region in which all periodic motions are locked by the external force. If the forcing remains moderate, this is the overlapping region for the leftmost and the rightmost Arnold tongues which correspond to the periodic orbits of the autonomous system with, respectively, the smallest and the largest values of ω_i. Inside this region the chaotic trajectories repeatedly visit the neighborhoods of the tori; moving along the surface of a

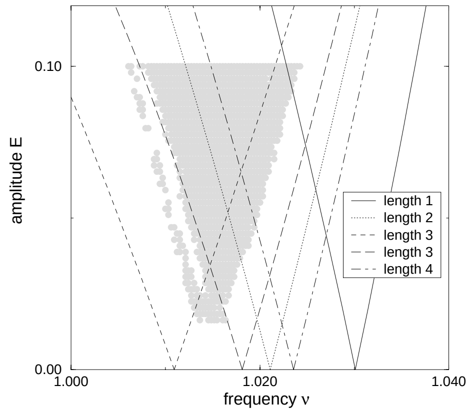

Figure 12.5: The Arnold tongues for the unstable periodic orbits in the Rössler system with different number of rotations around the origin. In the shadowed region the mean frequency of oscillations virtually coincides with the forcing frequency ν.

torus they approach the phase-stable solution and remain there for a certain time before the "transverse" (amplitude) instability bounces them to another torus. Since all periodic motions are locked, the phase remains localized within the bounded domain: one observes phase synchronization.

In Fig. 12.6 we show the phase portraits of the forced Rössler oscillator in the synchronized and non-synchronized states. The Poincaré maps are presented taken at the secant surface $y = 0$, the coordinates are the variable x of the Rössler system and the phase of the external force ψ (note that this representation is complementary to Fig. 12.4, where the phase of the external force is fixed). On these mappings the phase-stable orbits are represented by finite invariant subsets of points, they form a kind of the "skeleton" for the attractor. Similarly, phase-unstable orbits are a skeleton of the repeller (which is not shown in Fig. 12.6a); the latter plays a role of a barrier which separates the attraction domains of the two equivalent attractors whose phases differ by 2π. On approaching the boundary of the locked region from inside, the corresponding phase-stable and phase-unstable periodic orbits come closer. When they coalesce, attractor and repeller collide in the points of the "glued" orbit. After the bifurcation, a "channel" appears in the barrier, enabling phase slips during which the phase changes by $\pm 2\pi$. These slips appear in Fig. 12.6b as the rare points with the phases $\psi < 3$ and $\psi > 5.5$.

Since the Arnold tongues for different periodic orbits do not coincide, the onset of frequency lockings for these orbits occurs at different values of the frequency of external force. As a result, close to the threshold the synchronized segments of the trajectory alternate with the non-synchronized ones, and the whole transition to phase synchronization is smeared. The behavior observed at this transition is a specific kind of intermittency which we call "eyelet" since the seldom leakages from the locked state require the very precise hitting of certain small regions in the phase space.

The following description sketches the features of the transition mechanism (see [27]

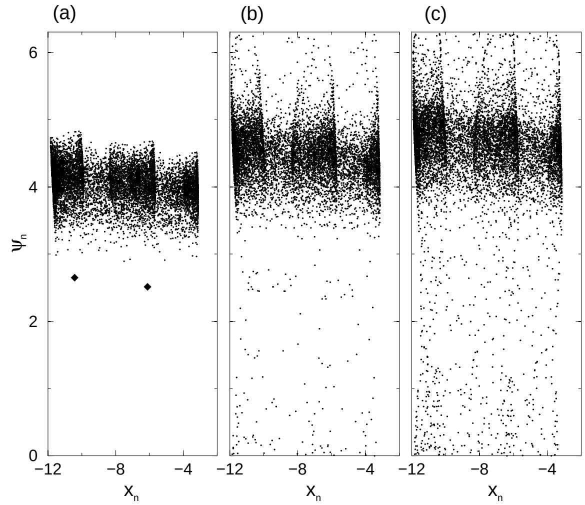

Figure 12.6: The Poincaré maps of the forced Rössler oscillator inside and outside the synchronization region. The markers denote the points belonging to the period-2 cycle, they lie apparently "outside" the attractor.

for the accurate derivation). The dynamics of the phase can be reasonably well approximated by the circle mapping. Just outside the tangent bifurcation the characteristic time intervals between the phase slips obey the inverse square root law: $\tau \approx C_1|\nu - \nu_c|^{-1/2}$ where ν_c is the bifurcation value of the frequency ν. Let the trajectory be reinjected into the vicinity of the respective unstable torus, at a small distance d_0 from it. To exhibit the slip, the trajectory should remain close to the torus for the time interval not smaller than τ. Within this time the distance to the torus grows: $d_\tau \approx d_0 e^{\lambda\tau}$ where λ is the positive Lyapunov exponent of the torus (for the weak forcing it is close to the Lyapunov exponent of the respective unstable periodic orbit in the autonomous system); we require d_τ to remain small: $d_\tau < C_2$. Assuming that the density of invariant probability on the attractor is (locally) uniform we estimate the probability to undergo a phase slip as proportional to the length of the interval d_0; for the latter holds

$$d_0 < C_2 \exp(-\lambda\tau) \approx C_2 \exp(-\lambda C_1 |\nu - \nu_c|^{-1/2}) \ .$$

Just outside the border of the synchronization region this interval is an extremely small "eyelet", and phase slips are exceptionally rare. In its turn, the increment δ of the rotation number (with respect to the value of ρ inside the locked region) is proportional to the averaged number of phase slips per mapping iteration, which leaves us with $\log\delta \approx -|\nu - \nu_c|^{-1/2}$ (Fig. 12.7).

The exponentially slow eyelet intermittency is the reason why the region of phase synchronization often appears to be larger than the overlapping part of the Arnold tongues and in certain cases seems to be observed also under small forcing amplitudes, for which there is no full phase synchronization at all. Only after a sufficiently large number of

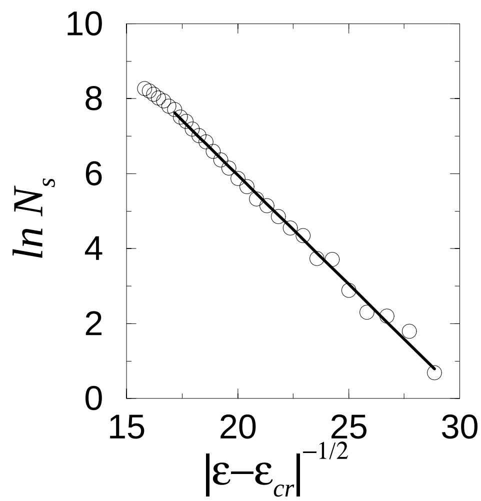

Figure 12.7: The average number of phase slips at the border of the synchronization region vs. the deviation of the forcing amplitude ε.

tangent bifurcations the probability of phase slip becomes noticeable, and one observes a deviation of the mean observed frequency from the frequency of the external force.

This picture is basically confirmed by numerical comparison of the domain of phase synchronization for the forced Rössler equations with the locked regions of individual periodic orbits (Fig. 12.5,12.7). However, certain peculiarities of the Rössler system do not fit the predictions. Phase synchronization is observed well below the intersection of the outermost tongues, i.e. in the domain where only a part of periodic orbits is synchronized. Thus, it appears that some periodic orbits do not contribute to the phase rotation. In the phase space, these orbits seem to lie outside of the bulk of the attractor (Fig. 12.6a); consequently, their vicinities are visited extremely seldom and possible phase slips are simply not detectable. Why some periodic orbits under the influence of forcing become "non-observable", remains an open question.

Analysis in terms of unstable periodic orbits allows one to understand the fine features of the onset of phase synchronization. We have discussed here the simplest case when the borders of the region of full phase synchronization are given by the phase-locking regions of the periodic orbits. More complex situations can occur if one of these borders is reached on a chaotic everywhere dense trajectory. Then the attractor and the repeller can collide in a dense set of points; similar situation is encountered in a quasiperiodically forced circle map [42, 43].

12.5 Phase synchronization in coupled systems

Now we demonstrate the effects of phase synchronization in coupled chaotic oscillators. We start with the simplest case of two interacting systems, and then briefly discuss oscillator lattices, globally coupled systems, and space-time chaos.

12.5.1 Synchronization of two interacting oscillators

We consider here two non-identical coupled Rössler systems

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2}, \\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{12.16}$$

where $a = 0.165$, $f = 0.2$, $c = 10$. The parameters $\omega_{1,2} = \omega_0 \pm \Delta\omega$ and ε determine the mismatch of natural frequencies and the coupling, respectively.

Again, like in the case of periodic forcing, we can define the mean frequencies $\Omega_{1,2}$ of oscillations of each system, and study the dependence of the frequency mismatch $\Omega_2 - \Omega_1$ on the parameters $\Delta\omega, \varepsilon$. This dependence is shown in Fig. 12.8 and demonstrates a large region of synchronization between two oscillators.

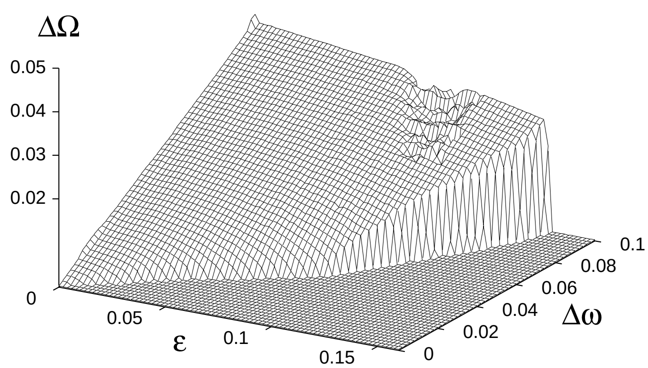

Figure 12.8: Synchronization of two coupled Rössler oscillators; $\omega_0 = 1$.

It is instructive to characterize the synchronization transition by means of the Lyapunov exponents (LE). The 6-order dynamical system (12.16) has 6 LEs (see Fig. 12.9). For zero coupling we have a degenerate situation of two independent systems, each of them has one positive, one zero, and one negative exponent. The two zero exponents correspond to the two independent phases. With coupling, the phases become dependent and the degeneracy must be removed: only one LE should remain exactly zero. We observe, however, that for small coupling also the second zero Lyapunov exponent remains extremely small (in fact, numerically indistinguishable from zero). Only at relatively stronger coupling, when the synchronization sets on, the second LE becomes negative: now the phases are dependent and a relation between them is stable. Note that the two positive exponents remain positive which means that the amplitudes remain chaotic and independent: the coupled system remains in the state of hyperchaos.

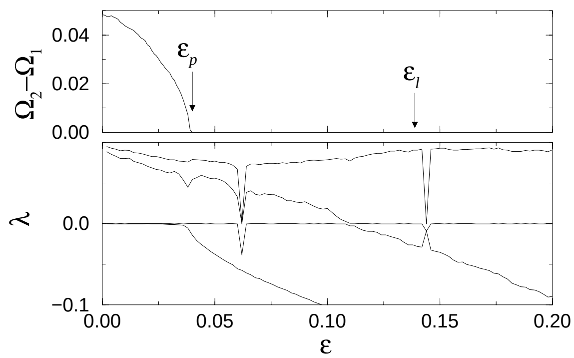

Figure 12.9: The Lyapunov exponents λ (bottom panel, only the 4 largest LEs are depicted) and the frequency difference vs. the coupling ε in the coupled Rössler oscillators; $\omega_0 = 0.97$, $\Delta\omega = 0.02$. Transition to the phase (ε_p) and to the lag synchronization (ε_l) are marked.

With the increase of coupling one of the positive LE becomes smaller. Physically this means that not only the phases are locked, but the difference between the amplitudes is suppressed by coupling as well. At a certain coupling only one LE remains positive, so one can expect synchronization both in phases and amplitudes. As the systems are not identical (due to the frequency mismatch), their states cannot be identical: $x_1(t) \neq x_2(t)$. However, almost perfect correspondence between the time-shifted states of the systems can be observed: $x_1(t) \approx x_2(t - \Delta t)$. This phenomenon is called “lag synchronization” [26]. With further increase of the coupling ε the lag Δt decreases and the states of two systems become nearly identical, like in case of complete synchronization (see the paper by Kocarev and Parlitz in this volume [14]).

12.5.2 Synchronization in a Population of Globally Coupled Chaotic Oscillators

A number of physical, chemical and biological systems can be viewed at as large populations of weakly interacting non-identical oscillators [35]. One of the most popular models here is an ensemble of globally coupled nonlinear oscillators (often called “mean-field coupling”). A nontrivial transition to self-synchronization in a population of periodic oscillators with different natural frequencies coupled through a mean field has been described by Kuramoto [35, 44]. In this system, as the coupling parameter increases, a sharp transition is observed for which the mean field intensity serves as an order parameter. This transition owes to a mutual synchronization of the periodic oscillators, so that their fields become coherent (i.e. their phases are locked), thus producing a macroscopic mean field. In its turn, this field acts on the individual oscillators, locking their phases, so that the synchronous state is self-sustained. Different aspects of this transition have been studied in [45, 46, 47], where also an analogy with the second–order phase transition has been exploited.

A similar effect can be observed in a population of *non-identical chaotic* systems, e.g.

the Rössler oscillators

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon X, \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= 0.4 + z_i(x_i - 8.5), \end{aligned} \tag{12.17}$$

coupled via the mean field $X = N^{-1} \sum_1^N x_i$. Here N is the number of elements in the ensemble, ε is the coupling constant, a and ω_i are parameters of the Rössler oscillators. The parameter ω_i governs the natural frequency of an individual system. We take a set of frequencies ω_i which are Gaussian-distributed around the mean value ω_0 with variance $(\Delta\omega)^2$. The Rössler system typically shows windows of periodic behavior as the parameter ω is changed; therefore we usually choose a mean frequency ω_0 in a way that we avoid large periodic windows. In our computer simulations we solve numerically Eqs. (12.17) for rather large ensembles $N = 3000 \div 5000$.

With an increase of the coupling strength ε, the appearance of a non-zero macroscopic mean field X is observed [22]. This indicates the phase synchronization of the Rössler oscillators that arises due to their interaction via mean field. This mean field is large, if the attractors of individual systems are phase-coherent (parameter $a = 0.15$) and the phase is well-defined. On the contrary, in the case of the funnel attractor $a = 0.25$, when the oscillations look wild, and the imaging point makes large and small loops around the origin, the field is rather small, and there seems to be no way to choose the Poincaré section unambiguously. Nevertheless, in both cases synchronization transition is clearly indicated by the onset of the mean field, without computation of the phases themselves.

12.6 Lattice of chaotic oscillators

If chaotic oscillators are ordered in space and form a lattice, only the nearest neighbors interact. Such a situation is relevant for chemical systems, where homogeneous oscillations are chaotic, and the diffusive coupling can be modeled with dissipative nearest neighbors interaction [48, 39]. In a lattice, one can expect complex spatio-temporal synchronization structures to be observed.

Consider as a model a 1-dimensional lattice of R"ossler oscillators with local dissipative coupling:

$$\begin{aligned} \dot{x}_j &= -\omega_j y_j - z_j, \\ \dot{y}_j &= \omega_j x_j + a y_j + \varepsilon(y_{j+1} - 2y_j + y_{j-1}), \\ \dot{z}_j &= 0.4 + (x_j - 8.5) z_j. \end{aligned} \tag{12.18}$$

Here the index $j = 1, \ldots, N$ counts the oscillators in the lattice and ε is the coupling coefficient. To study synchronization in a lattice of non-identical oscillators, we introduce a linear distribution of natural frequencies ω_j

$$\omega_j = \omega_1 + \delta(j-1) \tag{12.19}$$

where δ is the frequency mismatch between neighboring sites. Depending on the values of δ we observed two scenarios of transition to synchronization [23]. For small δ, the transition occurs smoothly, i.e. all the elements along the chain gradually adjust their frequencies. If the frequency mismatch is larger, clustering is observed: the oscillators build phase-synchronized groups having different mean frequencies. At the borders between clusters phase slips occur; this can be considered as appearance of defects in the spatio-temporal representation. Both regular and irregular patterns of defects have been reported in ref. [23].

12.7 Synchronization of space-time chaos

The idea of phase synchronization can be also applied to space-time chaos. E.g., in the famous complex Ginzburg-Landau equation [49, 50, 51]

$$\partial_t a = (1 + i\omega_0)a - (1 + i\alpha)|a|^2 a + (1 + i\beta)\partial_t^2 a \ , \tag{12.20}$$

there are regimes where the complex amplitude a rotates with some mean frequency, but these rotations are not regular: the phase deviates irregularly in space and time (this regime is called "phase turbulence"). Let us now add periodic in time spatially homogeneous forcing of amplitude B and frequency ω_e. Transition into a reference frame rotating with this external forcing ($a \to A \equiv a \exp(-i\omega_\mathrm{e} t)$) reduces Eq. (12.20) to

$$\partial_t A = (1 + i\nu)A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\partial_t^2 A + B \ , \tag{12.21}$$

where $\nu = \omega_0 - \omega_\mathrm{e}$ is the frequency mismatch between the frequency of the external force and the frequency of small oscillations. An analysis of different regimes in the system (12.21) has been recently performed [52]. As one can expect, a very strong force suppresses turbulence and the spatially homogeneous periodic in time synchronous oscillations are observed, while a small force has no significant influence on the turbulent state. A nontrivial regime is observed for intermediate forcing: in some parameter range the irregular fluctuations of the phase are not completely suppressed but are bounded: the whole system oscillates "in phase" with the external force and is highly coherent, although some small chaotic variations persist. One can easily see an analogy to the phase synchronization of chaotic oscillators, where chaos remains while the phase becomes entrained.

12.8 Detecting synchronization in data

The analysis of relation between the phases of two systems, naturally arising in the context of synchronization, can be used to approach a general problem in time series analysis. Indeed, bivariate data are often encountered in the study of real systems, and the usual aim of the analysis of such data is to find out whether two signals are dependent or not. As experimental data are very often non-stationary, the traditional techniques, such as cross–spectrum and cross–correlation analysis [53], or non–linear characteristics like generalized mutual information [54] or maximal correlation [55] have their limitations. From the other side, sometimes it is reasonable to assume that the observed signals originate from two weakly interacting systems. The presence of this interaction can be found by means of the analysis of *instantaneous* phases of these signals. These phases can be unambiguously obtained with the help of the analytic signal concept based on the Hilbert transform (for an introduction see [53, 24]). It goes as follows: for an arbitrary scalar signal $s(t)$ one can construct a complex function of time (analytic signal) $\zeta(t) = s(t) + i\tilde{s}(t) = A(t)e^{i\phi_H(t)}$ where $\tilde{s}(t)$ is the Hilbert transform of $s(t)$,

$$\tilde{s}(t) = \pi^{-1}\text{P.V.}\int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau \ , \tag{12.22}$$

and $A(t)$ and $\phi_H(t)$ are the instantaneous amplitude and phase (P.V. means that the integral is taken in the sense of the Cauchy principal value).

As recently shown in [19, 24], the phase defined by this method from an appropriately chosen oscillatory observable practically coincides with the phase of an oscillator computed according to one of the definitions given in Sec. 12.3. Therefore, the analysis of the relationship between these Hilbert phases appears to be an appropriate tool to detect synchronous epochs from experimental data and to check for a weak interaction between systems under study. It is very important that the Hilbert transform does not require stationarity of the data, so we can trace synchronization transitions even from nonstationary data.

We recall again the above mentioned similarity of phase dynamics in noisy and chaotic oscillators (see Sect. 12.3.2). A very important consequence of this fact is that, using the synchronization approach to data analysis, we can avoid the hardly solvable dilemma "noise vs chaos": irrespectively of the origin of the observed signals, the approach and techniques of the analysis are unique. Quantification of synchronization from noisy data is considered in [56].

Application of these ideas allowed us to find phase locking in the data characterizing mechanisms of posture control in humans while quiet standing [32, 28]. Namely, the small deviations of the body center of gravity in anterior–posterior and lateral directions were analyzed. In healthy subjects, the regulation of posture in these two directions can be considered as independent processes, and the occurrence of some interrelation possibly indicates a pathology. It is noteworthy that in several records conventional methods of time series analysis, i.e. the cross–spectrum analysis and the generalized mutual information failed to detect any significant dependence between the signals, whereas calculation of the instantaneous phases clearly showed phase locking.

Complex synchronous patterns have been found recently in the analysis of interaction of human cardiovascular and respiratory systems [33]. This finding possibly indicates the existence of a previously unknown type of neural coupling between these systems.

Analysis of synchronization between brain and muscle activity of a Parkinsonian patient [56] is relevant for a fundamental problem of neuroscience: can one consider the synchronization between different areas of the motor cortex as a necessary condition for establishing of the coordinated muscle activity? It was shown [56] that the temporal evolution of the coordinated pathologic tremor activity directly reflects the evolution of the strength of synchronization within a neural network involving cortical motor areas. Additionally, the brain areas with the tremor-related activity were localized from noninvasive measurements.

12.9 Conclusions

The main idea of this paper is to demonstrate that synchronization phenomena in periodic, noisy and chaotic oscillators can be understood within a unified framework. This is achieved by extending the notion of phase to the case of continuous-time chaotic systems. Because the phase is introduced as a variable corresponding to the zero Lyapunov exponent, this notion should be applicable to any autonomous chaotic oscillator. Although we are not able to propose a unique and rigorous approach to determine the phase, we have shown that it can be introduced in a reasonable and consistent way for basic models of chaotic dynamics. Moreover, we have shown that even in the case when the phases are not well-defined, i.e. they cannot be unambiguously computed explicitly, the presence of phase synchronization can be demonstrated indirectly by observations of the mean field and the spectrum, i.e. independently of any particular definition of the phase.

In a rather general framework, any type of synchronization can be considered as appearance of some additional order inside the dynamics. For chaotic systems, e.g., the complete synchronization means that the dynamics in the phase space is restricted to a symmetrical submanifold. Thus, from the point of view of topological properties of chaos, the synchronization transition usually means the simplification of the structure of the strange attractor. In discussing the topological properties of phase synchronization, we have shown that the transition to phase synchronization corresponds to splitting of the complex invariant chaotic set into distinctive attractor and repeller. Analogously to the complete synchronization, which appears through the pitchfork bifurcation of the strange attractor, one can say that the phase synchronization appears through tangent bifurcation of strange sets.

Because of the similarity in the phase dynamics, one may expect that many, if not all, synchronization features known for periodic oscillators can be observed for chaotic systems as well. Indeed, here we have described effects of phase and frequency entrainment by periodic external driving, both for simple and space-distributed chaotic systems. Further, we have described synchronization due to interaction of two chaotic oscillators, as well as self-synchronization in globally coupled large ensembles.

As an application of the developed framework we have discussed a problem in data analysis, namely detection of weak interaction between systems from bivariate data. The three described examples of the analysis of physiological data demonstrate a possibility to detect and characterize synchronization even from nonstationary and noisy data.

Finally, we would like to stress that contrary to other types of chaotic synchronization, the phase synchronization phenomena can happen already for very weak coupling, which offers an easy way of chaos regulation.

Acknowledgements

We thank G. Osipov, H. Chaté, O. Rudzick, U. Parlitz, P. Tass, C. Schäfer for useful discussions. A.P. and M.Z. acknowledge support of the Max-Planck-Society.

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