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Birth of strange nonchaotic attractors due to interior crisis

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Abstract

We study the interior crisis in the period-3-window of the quasiperiodically forced logistic map. Two routes from quasiperiodicity to chaos involving strange nonchaotic attractors (SNA) are discovered: Along one route we observe a sudden widening of the SNA. This is similar to the interior crisis in chaotic systems. Along the other route we find a direct transition from an invariant curve to a strange nonchaotic attractor exactly at the interior crisis point. This is a new mechanism of the appearance of strange nonchaotic attractors. Beyond the interior crisis the temporal behavior can be described as a crisis-induced intermittency, whose scaling behavior is discussed.

Keywords: Quasiperiodically forced systems; Interior crisis; Strange nonchaotic attractors

1. Introduction

Dynamical systems driven by external signals occur in many different fields of natural sciences. While periodically forced oscillators have been widely studied [1–3], quasiperiodically forced systems received much less interest. The latter can be realized either by maps where one of the maps is a rotation by an irrational multiple of 2π, or by ordinary differential equations driven at two incommensurate frequencies. Such quasiperiodically forced systems exhibit an interesting and unusual behavior related to a specific kind of attractors, the strange nonchaotic attractors (SNA) [4–7]. These attractors are strange in the geometrical sense, while they are nonchaotic in the dynamical sense. SNAs can be regarded as structures in between regularity and chaos since they possess properties which can be related either to regular or to chaotic processes. Corresponding to regular motion they do not show sensitivity with respect to changes in the initial conditions, i.e., all their Lyapunov exponents are negative (besides the zero related to the quasiperiodic forcing). However, typical trajectories experience arbitrarily long time intervals of expansion, while in the average contraction dominates [8–10]. This behavior yields a strange geometric structure in the phase space similar to those for chaotic attractors. SNA can also exhibit a singular continuous spectrum lying in between the pure point spectrum corresponding to regular (periodic or quasiperiodic) motion and the broad band spectrum characterizing chaotic motion.

Only for the model studied by Grebogi et al. [4] one can argue analytically that SNA exists and can prove rigorously its strange structure [11]. Numerical investigations have shown that attractors with a similar strange nonchaotic structure can be found in many different systems like the quasiperiodically forced logistic map [12,13], the quasiperiodically forced circle

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map [5,7] and the quasiperiodically forced pendulum [6,14]. They have been related to Anderson localization in the Schrödinger equation with a quasiperiodic potential [14]. The response of nerves on two signals with incommensurate frequencies can also be interpreted as a strange nonchaotic process [15]. Furthermore, the existence of SNA has been demonstrated in experiments with a quasiperiodically forced magnetoelastic ribbon [16] and in an oscillator with a multistable potential [17].

An important question is how SNAs can appear. Different mechanisms have been proposed for the emergence of SNA in quasiperiodically forced systems [7,12,18]. Most of them involve a collision between a stable and an unstable invariant curve in the case of maps, which corresponds to a collision of a stable and an unstable torus in the case of differential equations. For such mechanisms it can be shown that SNA can be explained as a complexly interwoven structure of a stable and an unstable solution [19]. In this paper we describe *the creation of SNA as a result of an interior crisis* which again involves an unstable saddle-type solution. We show that a region of parameters exist where the interior crisis does not lead to chaos, as one might expect, but yields the formation of an SNA. For chaotic systems it is known that the interior crisis is a mechanism for a sudden enlargement of a chaotic attractor, consisting formerly of different pieces [20–22]. For quasiperiodic systems in addition to the known mechanism, we obtain a transition either from a quasiperiodic motion or from a three-band SNA (before interior crisis) to a one-band SNA (after interior crisis). Thus, in contrast to the chaotic case, both attractors before and after the crisis are *nonchaotic* ones. In the case of a transition from quasiperiodicity the interior crisis results in the birth of an SNA, while in the case of the transition from a three-band SNA to a one-band one we observe only a sudden increase in the attractor size similar to that in chaotic systems.

The paper is organized as follows: As our basic model we introduce the quasiperiodically forced logistic map (Section 2) which possesses a strange nonchaotic attractor in different regions of the parameter plane. In Section 3 we describe briefly the properties of the ordinary logistic map in the vicinity of the period-3-window. We discuss in detail the behavior in the corresponding window for the quasiperiodically forced logistic map. In particular we consider the influence of the quasiperiodic force on the interior crisis and its characteristics. An important problem in the study of SNAs is their identification and distinction from nonstrange nonchaotic attractors. In Section 4 we recapitulate the special techniques that have been developed to detect them in model systems [8] and apply them to compute the transition points from quasiperiodicity to SNA. We study in detail the different types of interior crisis obtained for different routes in parameter space. Furhermore, we demonstrate that at the interior crisis the attractor collides with a chaotic saddle leading to an SNA. The temporal behavior following the interior crisis is determined by crisis-induced intermittency. The scaling behavior of the average times between bursts is quite similar to that known from chaotic systems. In Section 5 we summarize the results.

2. The quasiperiodically forced logistic map as the basic model

To study the appearance of SNA due to an interior crisis, we choose the quasiperiodically forced logistic map as our basic model:

$$x_{k+1} = r x_k (1 - x_k) + \epsilon \cos(2\pi\theta_k),$$
$$\theta_{k+1} = \theta_k + \omega \bmod 1 \qquad (1)$$

The first equation is the well-known logistic map influenced by an additive forcing. The parameter r determines the strength of the nonlinearity and ϵ models the amplitude of the forcing. The second equation describes the forcing as a shift on the circle which can be either periodic or quasiperiodic depending on ω. Rational ω means periodic forcing, while quasiperiodic forcing is caused by an irrational shift ω. Since we restrict ourselves to quasiperiodic forcing we use the inverse of the golden mean $\omega = (\sqrt{5} - 1)/2$ for all our computations below.

3. Crisis in the quasiperiodically forced logistic map

3.1. Properties of the period-3-window in the unforced logistic map

Interior crisis has been discovered first for the logistic map when studying the special bifurcations which are responsible for the appearance and disappearence of periodic windows [20]. Each periodic window is opened due to a saddle-node bifurcation creating a stable and an unstable period-p solution. The window is closed by an *interior crisis* where this unstable period-p solution collides with the chaotic attractor. The remarkable feature of this chaos to chaos transition is a sudden widening of the attractor as well as a simultaneous reduction of its pieces.

Interior crisis has been found likewise in more complicated models [22,23], as well as in experiments (e.g., [24,25]), i.e., this is a typical phenomenon occurring in nonlinear systems. The appearence of an interior crisis corresponds to the collision of the attractor with a chaotic saddle or, equivalently, with the basin boundaries of the p times iterated system. Furthermore, those basin boundaries were found to be very different structured, sometimes even fractal.

Next, we will focus our considerations to the period-3-window of the *quasiperiodically driven logistic map*.

3.2. The period-3-window in the quasiperiodically forced logistic map

If the logistic map is quasiperiodically forced as in Eq. (1), one has to study the interior crisis depending on two control parameters. Therefore it is necessary to find an extended definition of interior crisis which grasps the essential properties observed for crisis in 1-D systems.

As interior crisis we denote now a transition with the following properties:
- At the interior crisis there is a sudden widening of the attractor.

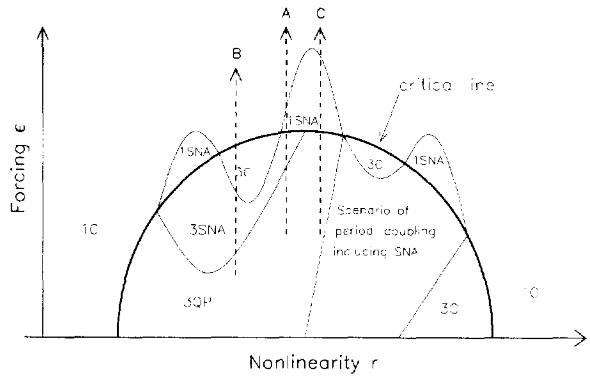

Fig. 1. Sketch of the structure of the period-3-window for the quasiperiodically forced logistic map depending on the two parameters nonlinearity r and forcing amplitude ϵ.

- Simultaneously, the number of attractor pieces is reduced.

It should be mentioned that an interior crisis need not be a transition from chaos to chaos. Further, in contrast to other definitions of interior crisis the notion of unstable periodic orbits or invariant curves is not used.

We restrict ourselves to those cases of interior crisis where a transition from a 3-piece attractor to a single-piece attractor appears. For each of the parameter values r, belonging to the period-3-window in the unforced system, exactly one value of the forcing parameter ϵ is found at which such an interior crisis appears. These points appear to be arranged on a continuous line in parameter space which we call critical line. We will study in more detail the transition phenomena along this critical line. These transitions comprise not only a change of the topological structure. Moreover, they sometimes represent a qualitative change in the dynamical properties of the system.

In Fig. 1 a schematic representation of the period-3-window is shown. The thick line represents the critical line. For zero forcing, the two equations are uncoupled and the behavior is equivalent to that of the original logistic map. But due to the enlarged state space all periodic orbits appear now as invariant curves. Therefore, at the parameters ($r = 1 + \sqrt{8}, \epsilon = 0$) a transition from a one-band chaotic attractor (1C) to

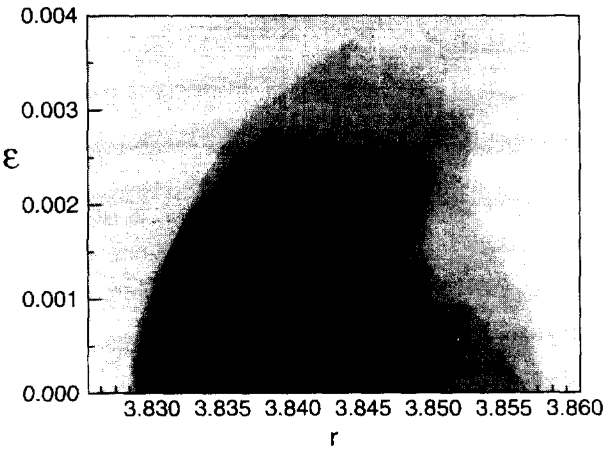

Fig. 2. Lyapunov exponents in the period-3-window for the quasiperiodically forced logistic map. The four dark grey areas correspond to negative and the four light grey areas to positive Lyapunov exponents.

a quasiperiodic motion with three-bands (3QP) takes place, corresponding to a saddle-node bifurcation. At ($r = 3.8568, \epsilon = 0$) a transition from a three-band chaotic attractor to a single-band attractor (3C $\longrightarrow$ 1C) occurs. The critical line studied here is connecting these two points in parameter space.

The whole bifurcation structure around the window is very complex (cf. Fig. 2) and contains several routes into chaos leaving the window. However, we focus on three typical transitions from quasiperiodic attractors (3QP) to chaotic one-band-attractors (1C) involving an interior crisis and the emergence of SNA. These routes are represented by vertical dashed lines in Fig. 1.

Route A. With increasing force the invariant curves (3QP) lose their smoothness and an SNA with three branches (3SNA) appears. This transition due to the loss of smoothness has been described for several model systems [10,13,26]. With further increase of the forcing we approach the crisis point, at which we observe a sudden change in the size of the attractor, this growth in size is connected with a transition from an SNA consisting of three pieces (3SNA) to an attractor consisting of one piece (1SNA). A transition to chaos (1C) occurs only for even higher forcing amplitudes. The scheme of the complete transitions can be denoted by

$$\text{3QP} \longrightarrow \text{3SNA} \underset{\substack{\uparrow \\ \text{interior crisis}}}{\longrightarrow} \text{1SNA} \longrightarrow \text{1C}$$

Such a route from quasiperiodicity to chaos can be observed for a fixed nonlinearity $r = 3.8335$ and varying forcing $\epsilon \in (0.00192, 0.0021)$. The corresponding attractors are represented in Fig. 3.

Route B. This route is similar to route A in the sense that it includes a transition from quasiperiodicity (3QP) to an SNA (3SNA) due to the loss of smoothness. But in contrast to route A the transition to chaos and the interior crisis change places so that we first observe the transition to chaos and then the interior crisis. The crisis involves here the three-piece chaotic attractor (3C) which turns into a one-piece chaotic attractor (1C), the well-known type of crisis which occurs in chaotic systems.

$$\text{3QP} \longrightarrow \text{3SNA} \longrightarrow \text{3C} \underset{\substack{\uparrow \\ \text{interior crisis}}}{\longrightarrow} \text{1C}$$

Such transitions occur in the considered system for $r = 3.8325$ and $\epsilon \in (0.0015, 0.002)$. Since this transition is well understood, we do not illustrate it in figures.

Route C. Along this route in parameter space we obtain a new mechanism for the emergence of SNA. In contrast to the routes A and B above there are no transitions to SNA before reaching the critical line along C. Here, SNA appears from the quasiperiodic motion as a result of the interior crisis exactly on the critical line.

$$\text{3QP} \underset{\substack{\uparrow \\ \text{interior crisis}}}{\longrightarrow} \text{1SNA} \longrightarrow \text{1C}$$

This type of transition can be found at $r = 3.836$ with $\epsilon \in (0.0024, 0.0025)$, the attractors are shown in Fig. 4.

From these three transitions only B is known and explained [20–22]. The interior crisis phenomena observed along the lines A and C are described and analyzed in the next section.

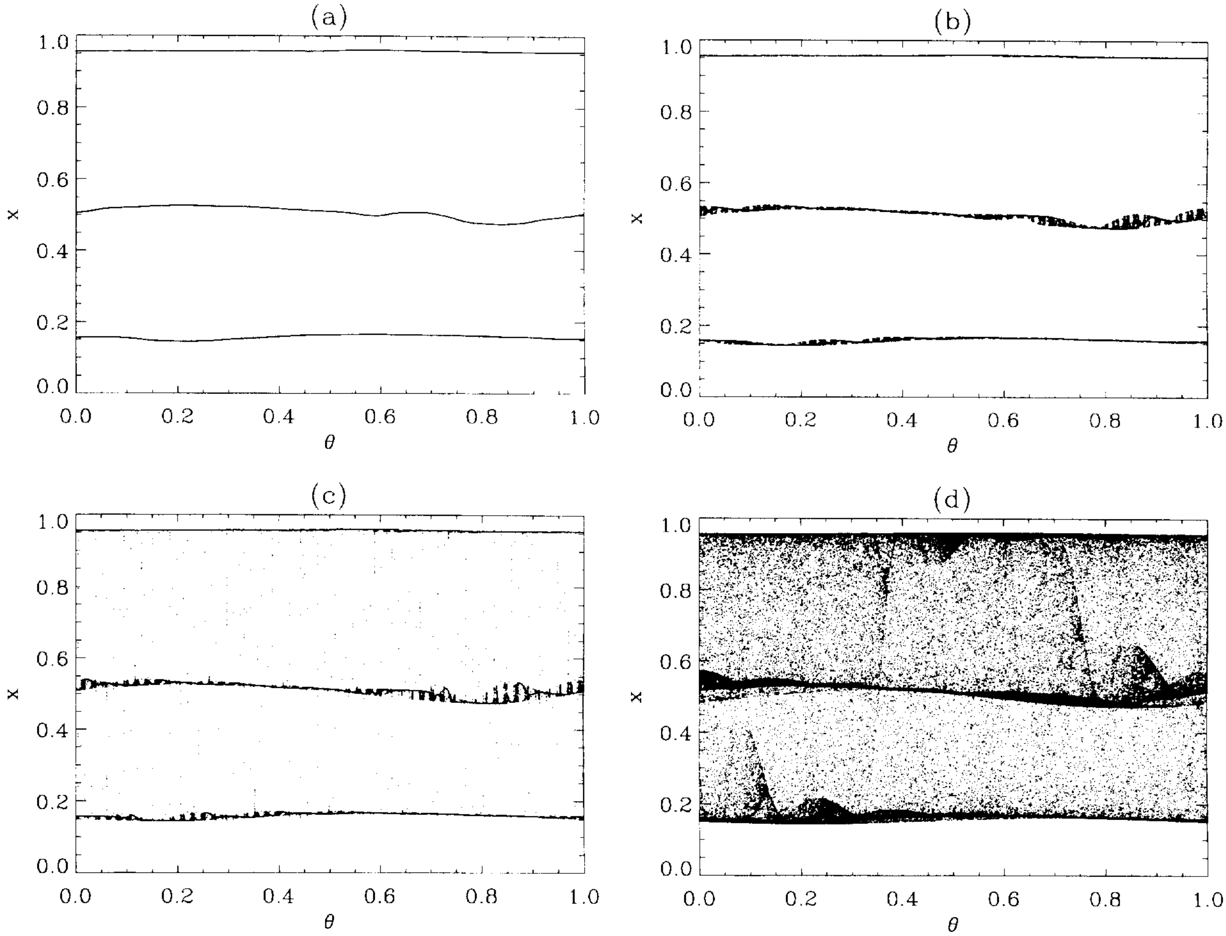

Fig. 3. Phase portraits of the attractors along Route A in Fig. 1 with $r = 3.8335$; (a) quasiperiodic motion with three branches (3QP) $\epsilon = 0.00192$; (b) strange nonchaotic attractor with three branches (3SNA) before interior crisis $\epsilon = 0.00196$; (c) strange nonchaotic attractor with one branch (1SNA) after interior crisis $\epsilon = 0.001995$; (d) chaotic attractor (1C) $\epsilon = 0.0021$.

4. The appearance of strange nonchaotic attractors

4.1. Methods to detect strange nonchaotic attractors

To describe the transition from an invariant curve to an SNA in detail it is necessary to develop methods to detect SNA in nonlinear systems of the form

$$x_{k+1} = f(x_k, \theta_k), \qquad \theta_{k+1} = \theta_k + \omega. \tag{2}$$

Our approach to check for the existence of SNA includes two different aspects. The easy first one is to show that the attractor is nonchaotic, i.e., one has to calculate the Lyapunov exponents. The more difficult second one is to show that the attractor is strange in the sense that it cannot be represented as a sufficiently smooth function $x = x(\theta)$, i.e., the attractor is not a smooth invariant curve. Two different methods have been proposed to distinguish between smooth and non-smooth attractors. We briefly recall the basic ideas of these two techniques, which are outlined in detail in [8] and which are used in this investigation to estimate the transition points from quasiperiodicity, where the attractor is a smooth invariant curve, to SNA.

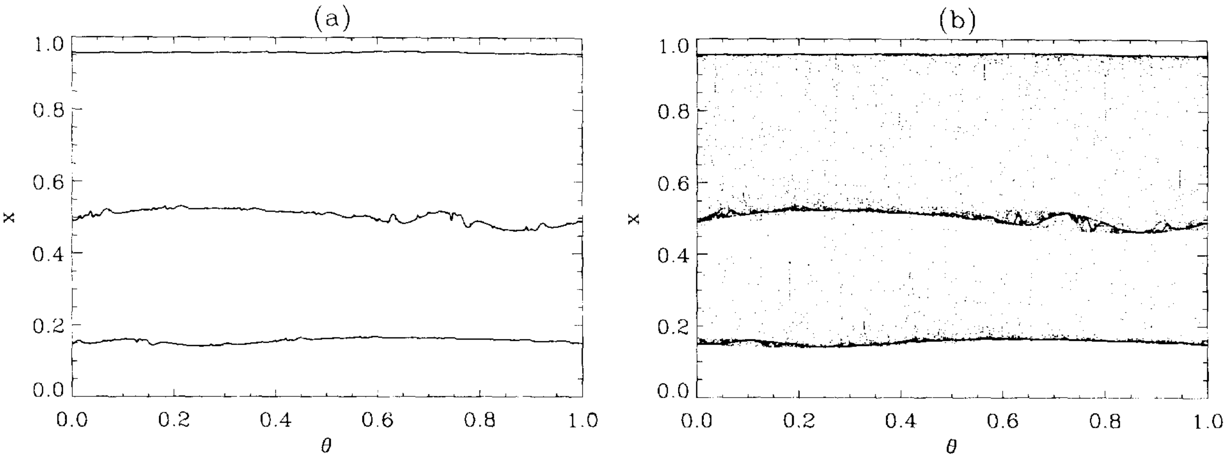

Fig. 4. Phase portraits of the attractors along Route C in Fig. 1 with $r = 3.836$; (a) quasiperiodic motion with three branches (3QP) before interior crisis $\epsilon = 0.002418$; (b) strange nonchaotic attractor with one branch (1SNA) after interior crisis $\epsilon = 0.00242$.

4.1.1. Rational approximations for the irrational driving frequency

This approach is based on the fact that every irrational number can be approximated by rationals using a continued fraction representation. Thus we replace the irrational number ω, which corresponds to a quasiperiodic forcing, by the rationals $\omega_n = p_n/q_n$, which yields a periodic forcing (p_n, q_n integers). For such periodically forced systems we observe different attractors depending on the initial phase θ_0. To obtain an approximation of the attractor of the quasiperiodically forced system, we change the initial phase θ_0 systematically within the interval [0, 1]. The union of all attractors for all used θ_0 gives us an approximation of the attractor for $\omega = \omega_n$, respectively. Since in Eq. (2) f is supposed to be a nonlinear function, different θ_0 may lead to different attractors. Thus we can consider the initial phase θ_0 as a bifurcation parameter. It turns out that the analysis of these bifurcation diagrams yields a criterion to distinguish between strange and nonstrange (smooth) attractors: The attractor is *strange* if we obtain with better and better approximation of the irrational ω, i.e., with increasing n, a *growing number of bifurcations* depending on θ_0. If there are *no bifurcations*, the attractor is *smooth*.

Since we use for our calculations the inverse of the golden mean $\omega = (\sqrt(5) - 1)/2$ as irrational forcing, the approximating rationals are then $\omega_n = p_n/q_n = F_n/F_{n+1}$, where F_n are the Fibonacchi numbers $F_n = 1, 1, 2, 3, 5, 8, \ldots$ If f is the logistic map the possible bifurcations are saddle-node bifurcations, period-doublings as well as boundary and interior crisis.

In Figs. 5 and 6 attractors for some rational approximants of the forcing are plotted. In the case of a quasiperiodic attractor and sufficiently good approximation of the forcing the resulting attractor does not contain bifurcations, it is a smooth line (Fig. 5). But, for an SNA one finds invariant sets which exhibit bifurcations for arbitrary good rational approximations (Fig. 6).

4.1.2. Sensitivity with respect to the external phase

SNAs are not sensitive with respect to changes in the initial conditions, which is reflected by the negative Lyapunov exponents. However, they exhibit a sensitivity with respect to the phase of the external force. This can be shown using a recurrence relation for the computation of the derivative with respect to the external phase $\partial x_n/\partial\theta$

$$\frac{\partial x_{n+1}}{\partial\theta} = f_\theta(x_n, \theta_n) + f_x(x_n, \theta_n)\frac{\partial x_n}{\partial\theta}, \qquad (3)$$

where f_θ and f_x denote the partial derivatives with respect to θ and x. To ensure that one calculates the

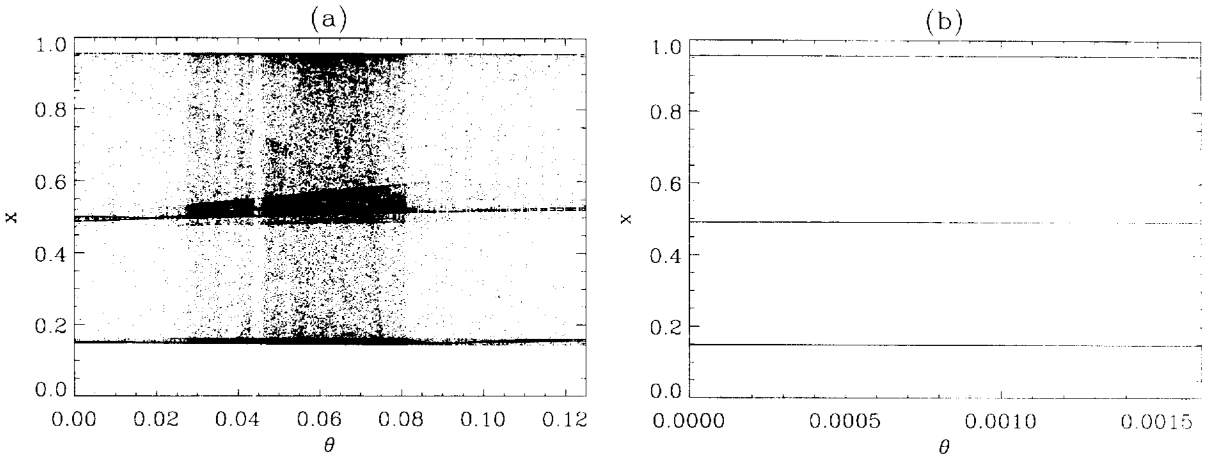

Fig. 5. Bifurcations in rational approximations for the smooth attractor before interior crisis ($r = 3.836$, $\epsilon = 0.002418$): (a) $\omega = 5/8$; (b) $\omega = 377/610$.

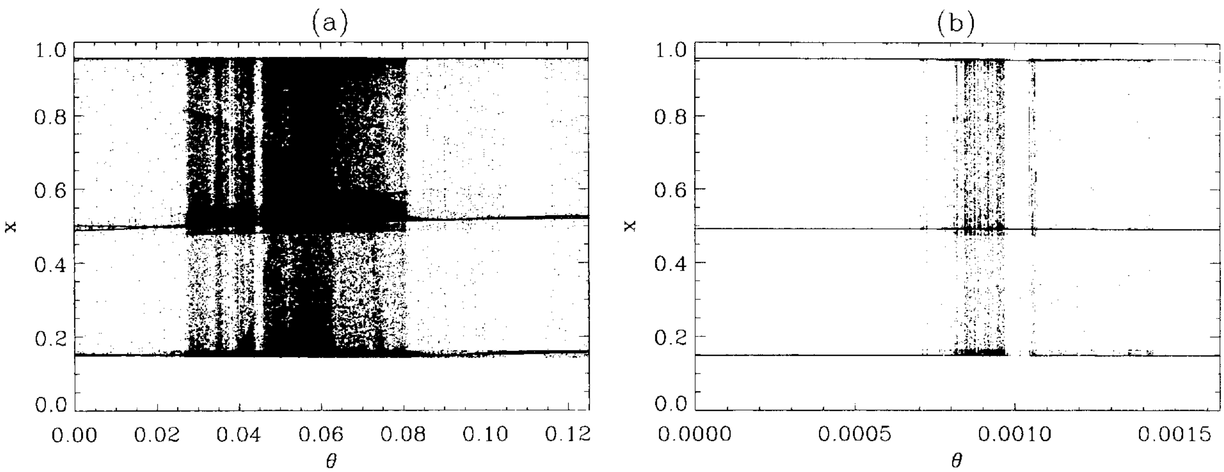

Fig. 6. Bifurcations in rational approximations for the strange nonchaotic attractor after interior crisis ($r = 3.836$, $\epsilon = 0.00242$): (a) $\omega = 5/8$; (b) $\omega = 377/610$.

derivative along the attractor one has to iterate (3) together with (2) starting from arbitrary values for x_0, θ_0 and $\partial x_0/\partial\theta$. The temporal behavior of $\partial x_n/\partial\theta$ is very intermittent, but looking only at the maximum

$$\gamma_N(x,\theta) = \min_{x_0,\theta_0} \max_{0\le n\le N} |\partial x_n/\partial\theta|, \tag{4}$$

yields another criterion to distinguish between strange and nonstrange attractors. In the case of a nonstrange (smooth) attractor γ_N grows up to the largest possible value of the derivative $|\partial x/\partial\theta|$ along the attractor and remains constant for all subsequent iterations as large one may choose N. In contrast, in the case of a strange attractor this quantity γ_N grows unboundedly and exhibits no saturation even for very large N (cf. Fig. 7). The growth of γ_N can be measured by a phase sensitivity exponent μ

$$\gamma_N = N^{\mu}. \tag{5}$$

This unbounded growth of the derivative with respect to the external phase can be explained in terms of local Lyapunov exponents, a concept which has been introduced to study the statistical properties of chaotic

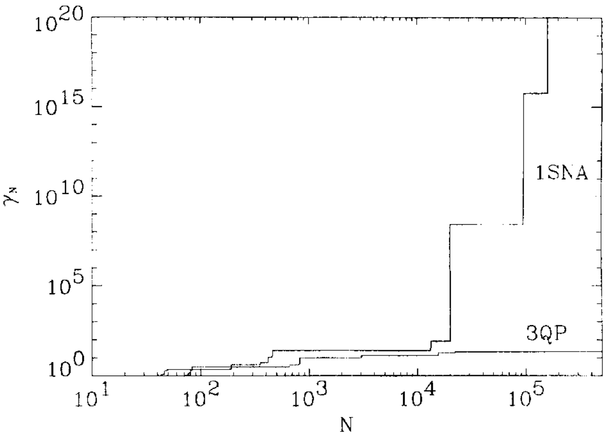

Fig. 7. Maximum value γ_N of the derivative with respect to the external phase $|\partial x/\partial\theta|$ $(r = 3.836)$.

systems (see, e.g., [27]). These local Lyapunov exponents are defined as finite time Lyapunov exponents

$$\Lambda_T = \frac{1}{T}\log\prod_{i=1}^{T}|f_x(x,\theta)|, \tag{6}$$

giving the usual Lyapunov exponent λ in the limit $\lambda = \lim_{T\to\infty}\Lambda_T$. The distribution of these local Lyapunov exponents for large ensembles of trajectories and large T scales as

$$\mathrm{prob}(\Lambda_T = \Lambda) \sim \exp(-T\Phi(\Lambda)). \tag{7}$$

This distribution $\Phi(\Lambda)$ has a maximum at negative Λ, but a nonvanishing tail in the positive region of Λ. Therefore, even if the averaged Lyapunov exponent is negative corresponding to nonchaotic behavior, there are arbitrary large stretches of the trajectory exhibiting a positive local Lyapunov exponent related to expansion on this time period. These arbitrary long time intervals of expanding behavior are closely related to the unbounded growth of $|\partial x/\partial\theta|$ and thus responsible for the strange structure of the attractor. Additionally this explains why SNAs in many cases occur in the neighborhood of the transition to chaos. But the existence of SNA is not only a transition phenomenon, they can exist for some parameter interval before chaos (cf. Fig. 8). The critical value for the interior crisis marked in Fig. 8 indicates also a characteristic change in the shape of the dependence of the Lyapunov exponents vs. the amplitude of the forcing. As soon as SNAs occur as a result of the crisis, the Lyapunov exponents vary not smoothly with the parameter as it is seen for the forcing before crisis.

4.2. *How does interior crisis occur?*

Several types of crisis have been studied in detail for chaotic systems. In the case of an interior crisis a sudden increase in the size of the chaotic attractor takes place as the parameter passes through the critical value. This change in size is due to a collision between the chaotic attractor and a chaotic saddle already existent when the crisis occurs [22]. As a result

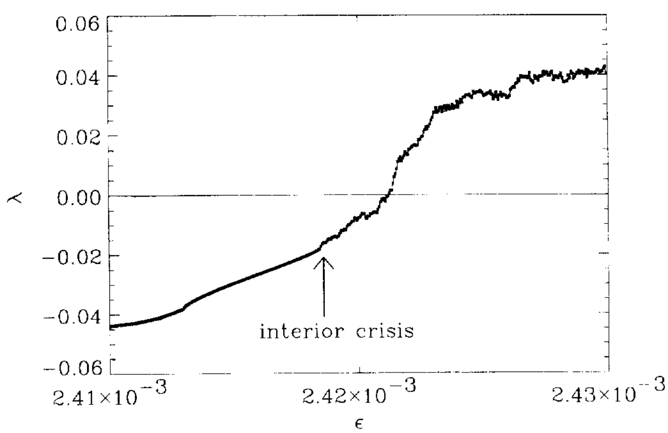

Fig. 8. Maximum Lyapunov exponent depending on the amplitude of forcing ϵ along Route C in Fig. 1 $(r = 3.836)$.

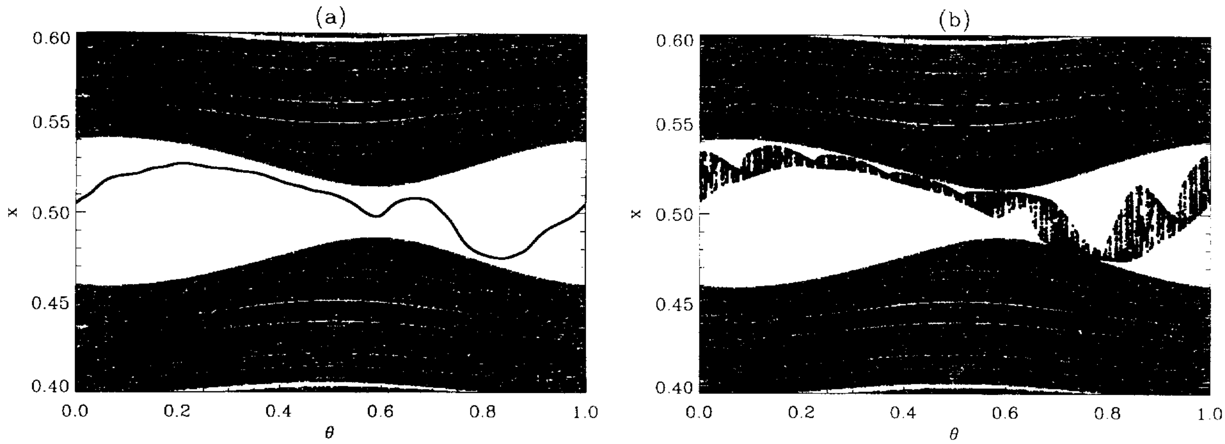

Fig. 9. Collision of the attractor with its basin boundary for one of the three attractors of the third iterate of map (1) along Route A in Fig. 1 ($r = 3.8335$). White dots denote points in the basin of the shown attractor, while black dots indicate points of the two other basins. (a) Far from interior crisis $\epsilon = 0.00192$; (b) just before interior crisis $\epsilon = 0.00196$.

the new attractor includes the former attractor pieces and the former chaotic saddle.

In the following we argue that the two new interior crisis phenomena in quasiperiodically forced systems (Routes A and C) are caused by the same reasons as in chaotic systems (Route B). But there are also important differences between interior crisis where chaotic attractors are involved and interior crisis in quasiperiodically forced systems. As already mentioned in the previous section the attractors before and after the interior crisis are *nonchaotic* ones. Before the crisis the attractor is either quasiperiodic (Route C) or strange nonchaotic (Route A). The result after the crisis is an SNA with one branch in both cases.

First, we demonstrate the possibility of a collision with a chaotic saddle. It is known that when the period-3-window opens, the chaotic attractor before the window is transformed into a chaotic saddle that exists throughout the whole window. But instead of a direct computation of this chaotic saddle we use the concept of fractal basin boundaries to illustrate the existence of this saddle. At the opening of the window there is a saddle-node bifurcation for two invariant curves with three branches, a stable and an unstable one. If we consider only every third iterate of Eqs. (1), then our map possesses three different attractors, which are invariant curves with one branch. Choosing a grid of initial conditions in the x–θ plane, we can compute the basins of attraction of each of these attractors and visualize the structure of these basins. In Figs. 9 and 10 the white dots denote points belonging to the basin of the plotted attractor, while the black dots mark the union of the basins of the two other, not plotted attractors. We obtain seemingly fractal basins in all considered cases, only their structure is different. For Route C the basins are more complex and contain islands of other basins due to the noninvertibility of the map (Figs. 9(a) and 10(a)). Fig. 9(a) shows a part of Fig. 3(a) containing only the "middle" branch of the complete attractor. As we approach the crisis point with increasing ϵ, the attractor (SNA in Route A and an invariant curve in Route C) gets closer to the basin boundary (Figs. 9(b) and 10(b); compare also with Fig. 3(b) and Fig. 4(a) containing the complete attractors). Finally it collides with the basin boundary, or more precisely with a chaotic saddle on it, to form a one branch SNA (Figs. 3(c) and 4(b)). Therefore, two different phenomena can occur at the crisis point: (1) along the route A at the interior crisis we observe a sudden increase in the size of SNA, which after the crisis includes the former pieces of the SNA and the saddle-type chaotic set; (2) along the route C the crisis leads to the emergence of the 1SNA from the collision of the quasiperiodic motion with the saddle.

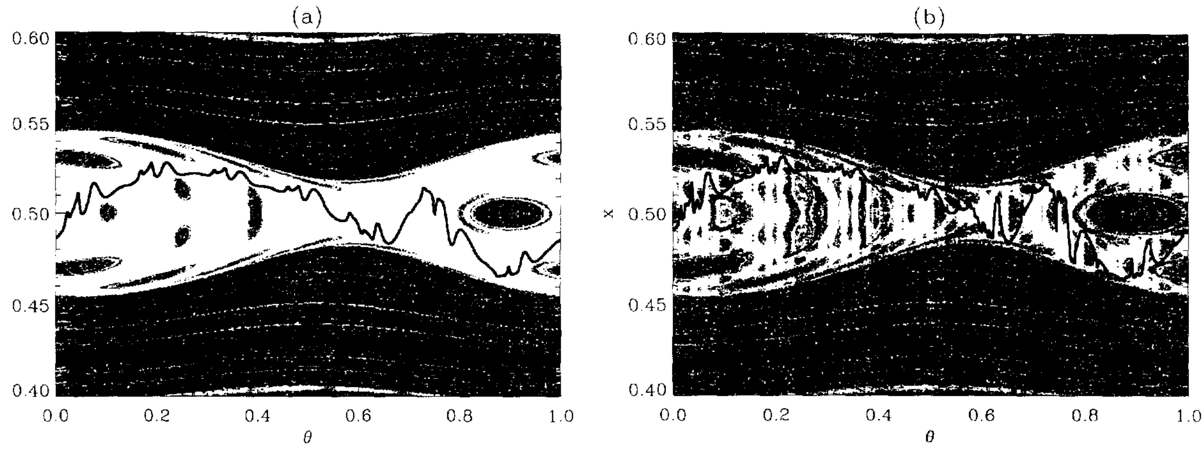

Fig. 10. Collision of the attractor with its basin boundary for one of the three attractors of the third iterate of map (1) along Route C in Fig. 1 ($r = 3.836$). White dots denote points in the basin of the shown attractor, while black dots indicate points of the two other basins. (a) Far from the interior crisis $\epsilon = 0.0023$; (b) just before interior crisis $\epsilon = 0.002418$.

4.3. *Characteristics of crisis-induced intermittency*

The temporal behavior after the interior crisis can be characterized as crisis-induced intermittency. A trajectory of the third iterate of Eqs. (1) spends some long stretch of time in the vicinity of one of the former attractors, then it bursts out from this region and bounces around in the region of the former chaotic saddle until it comes close to the same or another former attractor where it remains again for some time interval. This way the trajectory is irregularly jumping between the three former attractors as time goes to infinity. This temporal behavior, which can be observed for different types of crisis, can be described by a specific time τ corresponding to the type of the crisis [28]. For the interior crisis this time τ is defined to be the average over a long orbit of the time between bursts. The time intervals between two bursts themselves seem to be more or less random. But their average value τ possesses a scaling law as the parameter p approaches the critical value ϵ_c where the crisis occurs:

$$\tau \sim (\epsilon - \epsilon_c)^{-\gamma}. \tag{8}$$

We have checked whether such a power law behavior can be obtained also for the interior crisis in quasiperiodically forced systems. Indeed we find a power law (Fig. 11 for $r = 3.84$). The critical exponent γ for this parameter value has been determined as $\gamma \sim 0.5$. In general, the determination of the scaling behavior turned out to be rather difficult since the scaling regions for fitting the exponent γ are rather small, in most cases much smaller as the one shown in Fig. 11. Therefore, the influence of the quasiperiodic forcing on the scaling exponent is not so obvious and needs further investigation.

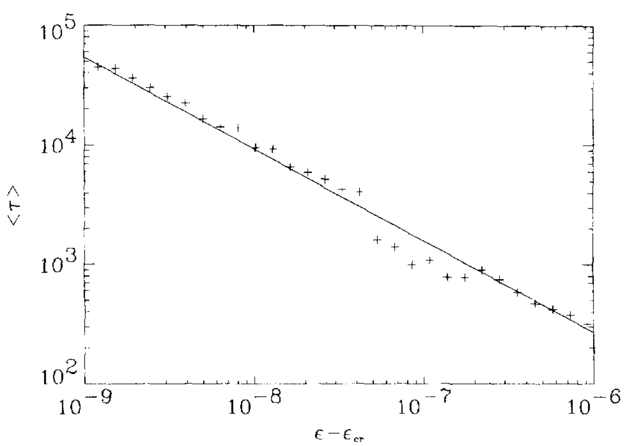

Fig. 11. Log–log plot of the average time between bursts vs. the distance from the critical line.

5. Summary

We have studied the interior crisis in the period-3-window in the quasiperiodically forced logistic map. Besides the well-known type of interior crisis in chaotic systems we have found two additional types which involve SNAs. In one type we obtain a sudden change in the size of the attractor as a result of the interior crisis. This is similar to the known mechanism for chaos. A three-piece SNA changes suddenly into a one-piece SNA. In the second type of interior crisis the transition from a smooth invariant curve to a SNA occurs. This is a new mechanism for the appearance of SNA. Thus, we can conclude that the interior crisis is a phenomenon which appears not only for chaotic attractors but also for nonchaotic ones. In both types of crisis, the attractor before and after the crisis is nonchaotic.

The interior crisis can be, generally, described as a collision between the attractor and a saddle-type invariant set. This can be shown by computing and visualizing the basins of attraction of the three attractors arising from the map of the third iterate. These basins have a complicated structure and their boundaries show some kind of metamorphosis transition [29], whose details will be reported elsewhere [30].

The temporal behavior after the crisis can be described as a crisis-induced intermittency. The scaling of the characteristic times between bursts scales as a power law with the distance from the crisis point.

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References

[1] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical systems, and Bifurcations of Vector Fields (Springer, Berlin, 1986).
[2] F.C. Moon, Chaotic Vibration. An Introduction for Applied Scientists and Engineeres (Wiley, New York, 1987).
[3] T. Kapitaniak, Chaotic Oscillations in Mechanical Systems (Manchester University Press, Manchester, 1991).
[4] C. Grebogi, E. Ott, S. Pelikan and J.A. Yorke, Physica D 13 (1984) 261.
[5] M. Ding, C. Grebogi and E. Ott, Phys. Rev. A 39 (1989) 2593.
[6] F.J. Romeiras and E. Ott, Phys. Rev. A 35 (1987) 4404.
[7] U. Feudel, J. Kurths and A. Pikovsky, Physica D 88 (1995) 176.
[8] A.S. Pikovsky and U. Feudel, Chaos 5 (1995) 253.
[9] Y.-C. Lai, Phys. Rev. E 53 (1996) 57.
[10] V.S. Anishchenko, T.E. Vadivasova and O. Sosnovtseva, Phys. Rev. E 53 (1996) 4451.
[11] G. Keller, Fundamenta Mathematicae (1995), in press.
[12] J.F. Heagy and S.M. Hammel, Physica D 70 (1994) 140.
[13] T. Nishikawa and K. Kaneko, Phys. Rev. E 54 (1996) 6114.
[14] A. Bondeson, E. Ott and T.M. Antonsen, Phys. Rev. Lett. 55 (1985) 2103.
[15] M. Pagitsas, A. Karantonis and D. Sazou, Int. J. Bifurc. and Chaos 2 (1992) 295.
[16] W.L. Ditto et al., Phys. Rev. Lett. 65 (1990) 533.
[17] T. Zhou, F. Moss and A. Bulsara, Phys. Rev. A 45 (1992) 5394.
[18] T. Yalcinkaya and Y.-C. Lai, Phys. Rev. Lett. 77 (1996) 5039.
[19] J. Stark, Physica D (1997), to appear.
[20] C. Grebogi, E. Ott and J.A. Yorke, Phys. Rev. Lett. 48 (1982) 1507.
[21] C. Grebogi, E. Ott and J.A. Yorke, Physica D 7 (1983) 181.
[22] Y.-C. Lai, C. Grebogi and J.A. Yorke, in: Applied Chaos, eds. J.H. Kim and J. Stringer (Wiley, New York, 1992).
[23] J.A.C. Gallas, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. 71 (1993) 1359.
[24] W.L. Ditto et al., Phys. Rev. Lett. 63 (1989) 923.
[25] J.C. Sommerer et al., Phys. Lett. A 153 (1991).
[26] O. Sosnovtseva, U. Feudel, J. Kurths and A. Pikovsky, Phys. Lett. A 218 (1996) 255.
[27] P. Grassberger, R. Badii and A. Politi, J. Stat. Phys. 51 (1988) 135.
[28] C. Grebogi, E. Ott, F. Romeiras and J.A. Yorke, Phys. Rev. A 36 (1987) 5365.
[29] C. Grebogi, E. Ott and J.A. Yorke, Phys. Rev. Lett. 57 (1986) 1284.
[30] U. Feudel, C. Grebogi, Y.-C. Lai and A. Witt, in preparation.