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Analysing Synchronization Phenomena from Bivariate Data by Means of the Hilbert Transform

Michael Rosenblum[1,2] and Jürgen Kurths[1]

[1] Max-Plank-Arbeitsgruppe "Nichtlineare Dynamik" an der Universität Potsdam, Am Neuen Palais 19, PF 601553, D-14415 Potsdam, Germany
[2] A. von Humboldt Fellow. Permanent address: Mechanical Engineering Research Institute, Russian Academy of Sciences, 101830 Moscow, Russia

Abstract. We use the analytic signal approach based on the Hilbert transform to compute the phase difference between two non-stationary signals and find out epochs of phase locking.

1 Introduction

Bivariate data are often encountered in the study of physiological systems. The usual problem in the analysis of these data is whether two signals are dependent or not. As the data are practically always non-stationary, the application of traditional techniques such as cross-spectrum and cross-correlation analysis (Panter 1965) or nonlinear characteristics like generalized mutual information (Pompe 1993) has its limitations.

Another common problem occurs when the signals remind of periodic functions with slowly varying parameters. The natural approach here is to consider two time series as an output of two coupled oscillators, and to quantify their interaction by measuring the phase difference between these signals. As examples we can mention studies of coordinated movements (Fuchs et al. 1996; Tass et al. 1995) and cardiorespiratory interaction (Schieck 1994). Nevertheless, this procedure seems to be not trivial for non-sinusoidal signals (see discussion in Fuchs et al. 1996), and different *ad hoc* methods are used for phase calculation.

In the present work we would like to attract the attention to the *analytic signal approach* based on the Hilbert transform. This technique, widely used in the signal processing (Panter 1965; Rabiner and Gold 1975; Smith and Mersereau 1992), allows one to obtain unambiguously the phase difference for arbitrary signals. Presence of a certain relationship between phases is an indicator of some dependency between components of bivariate data. Thus, this method addresses both problems outlined above. As the Hilbert transform does not require stationarity of the data, variations of that dependency with time can be easily studied.

We relate the discussed method to the phenomenon of *phase synchronization of chaotic systems* recently demonstrated by Rosenblum et al. (1996).

Examples of application of the presented technique to the study of posture control data, visually guided forearm tracking, and interaction of cardiac and respiratory systems are given by Rosenblum et al. (this volume), Tass et al. (1996), and Hoyer et al. (this volume).

2 Instantaneous Phase of a Signal

A consistent way to define the phase of an *arbitrary* signal is known in signal processing as analytic signal concept (Panter 1965; Rabiner and Gold 1975; Smith and Mersereau 1992). This general approach, based on the Hilbert transform and originally introduced by Gabor (1946), unambiguously gives the *instantaneous phase and amplitude* for a signal $s(t)$ via construction of the *analytic signal* $\zeta(t)$, which is a complex function of time defined as

$$\zeta(t) = s(t) + j\tilde{s}(t) = A(t)e^{j\phi(t)} , \tag{1}$$

where the function $\tilde{s}(t)$ is the Hilbert transform of $s(t)$

$$\tilde{s}(t) = \pi^{-1}\text{P.V.}\int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau \tag{2}$$

and P.V. means that the integral is taken in the sense of the Cauchy principal value. The instantaneous amplitude $A(t)$ and the instantaneous phase $\phi(t)$ of the signal $s(t)$ are thus uniquely defined from (1).

As one can see from (2), the Hilbert transform $\tilde{s}(t)$ of $s(t)$ can be considered as the convolution of the functions $s(t)$ and $1/\pi t$. Due to the properties of convolution, the Fourier transform $\widetilde{S}(\omega)$ of $\tilde{s}(t)$ is the product of the Fourier transforms of $s(t)$ and $1/\pi t$. For physically relevant frequencies $\omega > 0$, $\widetilde{S}(\omega) = -jS(\omega)$. This means that the Hilbert transform can be realized by an ideal filter whose amplitude response is unity, and phase response is a constant $\pi/2$ lag at all frequencies (Panter 1965).

A harmonic oscillation $s(t) = A\cos\omega t$ is often represented in the complex notation as $A\cos\omega t + jA\sin\omega t$. It means that the real oscillation is complemented by the imaginary part which is delayed in phase by $\pi/2$, that is related to $s(t)$ by the Hilbert transform. The analytic signal is the direct and natural extension of this technique, as the Hilbert transform performs the $-\pi/2$ phase shift for every frequency component of an arbitrary signal.

An important advantage of the analytic signal approach is that the phase can be easily obtained from experimentally measured scalar time series. Numerically, this can be done via convolution of the experimental data with a pre-computed characteristic of the filter (Hilbert transformer) (Rabiner and Gold 1975; Smith and Mersereau 1992; Little and Shure 1992). Although Hilbert transform requires computation on the infinite time scale, i.e. Hilbert transformer is an infinite impulse response filter, the acceptable precision of about 1% can be obtained with the 256-point filter characteristic. The sampling rate must be chosen in order to have at least 20 points per average

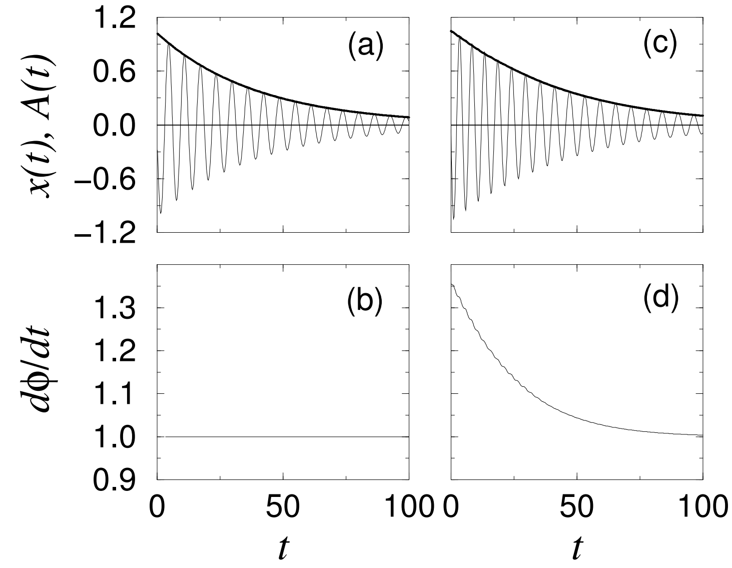

Fig. 1. Free vibrations $x(t)$ of the linear (a) and nonlinear (Duffing) (c) oscillators. The instantaneous amplitudes $A(t)$ calculated via Hilbert transform are shown by thick lines. Corresponding instantaneous frequencies $d\phi/dt$ are shown in (b) and (d)

period of oscillation. In the process of computation of the convolution $L/2$ points are lost at the both ends of the time series, where L is the length of the transformer.

We illustrate the properties of the Hilbert transform by the following examples.

Example 1. Harmonic oscillator. The Hilbert transform of the harmonic oscillation $x(t) = A\cos\omega t + \phi_0$ equals $\tilde{x}(t) = A\sin\omega t + \phi_0$; respectively the phase $\phi(t) = \omega t + \phi_0$. It means, that the *phase portrait* of the harmonic oscillator in coordinates $(x, \tilde{x})$ is a circle for any ω. Note, however, that the often used coordinates $(x, \dot{x})$ and delay coordinates $(x(t), x(t-\tau))$ generally produce an ellipse; more important, the phase obtained from such plots demonstrate oscillations that are the artifact of calculation (compare with the discussion in Fuchs et al. 1996).

Example 2. Damped oscillators. Let us take as the measured signals free oscillations of linear

$$\ddot{x} + 0.05\dot{x} + x = 0 \tag{3}$$

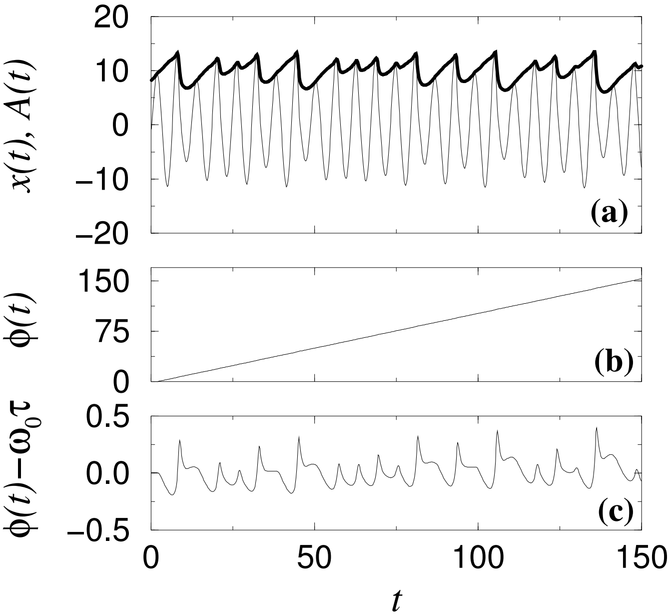

Fig. 2. Solution of the Rössler system $x(t)$ and its instantaneous amplitude $A(t)$ (thick line) (a). Instantaneous phase ϕ grows practically linear (b), nevertheless small irregular fluctuations are seen (c)

and Duffing

$$\ddot{x} + 0.05\dot{x} + x + x^3 = 0 \tag{4}$$

oscillators, and calculate from $x(t)$ instantaneous amplitudes $A(t)$ and frequencies $d\phi/dt$ (Fig. 1). The amplitudes, shown as thick lines, are really envelopes of decaying processes. The frequency of the linear oscillator is constant, while frequency of the Duffing oscillator is amplitude-dependent, as expected. Note, that although only about 20 periods of oscillations have been used, the nonlinear properties of the system can be easily seen from the time series, because frequency and amplitude are estimated in every point of the signal. This method is used in mechanical engineering for identification of elastic and damping properties of a vibrating system (Feldman 1985; Feldman and Rosenblum 1988; Feldman 1994).

Example 3. Rössler oscillator. Let us choose as an observable the x coordinate of the Rössler system

$$\begin{aligned} \dot{x} &= -y - z \, , \\ \dot{y} &= x + 0.15y \, , \\ \dot{z} &= 0.2 + z(x - 10) \, . \end{aligned} \tag{5}$$

Instantaneous amplitude and phase are shown in Fig. 2. The phase ϕ grows practically linear, nevertheless small irregular fluctuations of that growth are seen. This agrees with the known fact that oscillations of the system are chaotic, but the power spectrum of $x(t)$ contains a very sharp peak (Crutchfield et al. 1980).

3 Phase Synchronization of Chaotic Systems

Synchronization of periodic self-oscillatory systems is defined as a phase entrainment

$$|n\phi(t) - m\psi(t)| < \text{const} , \tag{6}$$

where n and m are integer numbers. In the presence of noise the phase difference is unbounded and performs a random-walk-like motion. However, if the noise is small, the frequencies are nearly locked, i.e. the relation between them is fulfilled in average:

$$n\langle\frac{d\phi}{dt}\rangle = m\langle\frac{d\psi}{dt}\rangle . \tag{7}$$

Phase synchronization of chaotic oscillators (Rosenblum et al. 1996; Pikovsky et al. 1997; Parlitz et al. 1996) is a direct generalization of this classical phenomenon. In the synchronous regime, the phases of interacting chaotic systems become locked, while the amplitudes vary chaotically, and are practically uncorrelated. A weaker type of synchronization has been also demonstrated, where the frequencies are entrained, while the phase difference exhibits a random-walk-type motion. Mutual phase synchronization of two nonidentical chaotic systems has been considered in Rosenblum et al. 1996a. It has been shown, that phase synchronization manifests itself in the Lyapunov spectrum of the coupled system: when the phase locking occurs, one of two zero Lyapunov exponents becomes negative.

The central problem in the study of phase synchronization is to introduce the notion of phase for chaotic oscillating system. There exist no unambiguous and strict definition. Nevertheless, often we can find a projection of the attractor on some plane (x, y) such that the plot reminds us of the smeared limit cycle, i.e. the trajectory rotates around the origin, or any other point that can be chosen as the origin. It means that we can choose the Poincaré section in a proper way. With the help of the Poincaré map we can define a phase, attributing 2π increase to each intersection of the trajectory with the secant surface. If the above mentioned projection is found, we can also introduce the phase as the angle between the projection of the phase point on the plane and a given direction on the plane, i.e. $\varphi = \arctan(y/x)$.

Another possibility is to calculate the instantaneous phase by taking some coordinate of the oscillating system as an observable. Although the analytic signal approach provides the unique determination of the phase of *a signal*, we cannot avoid ambiguity defining the phase for a *dynamical system*, as the result depends on the choice of the observable. Here we face the same problem as in the choice of the appropriate projection mentioned above. However, one can often find an "oscillatory" observable that provides the Hilbert phase ϕ_H in good agreement with our intuition. For example, the z-coordinate is a natural choice for the well-known Lorenz system. The detailed discussion of different definitions of the phase of the system can be found in Pikovsky et al. (1997). For the experimental studies, the phase calculated from the Hilbert transform is mostly convenient.

It is noteworthy that the phenomenon of phase synchronization is observed even when completely different systems, such as the Rössler oscillator and the Mackey–Glass differential-delay system, or the Rössler and the hyperchaotic Rössler oscillators, interact. Phase synchronization even occurs if the systems are qualitatively different, i.e. one is chaotic and another one periodic. Another important feature is that the phase synchronization is observed already for extremely weak coupling, and in some cases can have no threshold, contrary to other types of synchronization of chaotic systems.

4 Calculating Relative Phase

The relative phase, or phase difference of two signals $s_1(t)$ and $s_2(t)$ can be obtained via the Hilbert transform as

$$\varphi_1(t) - \varphi_2(t) = \arctan \frac{\tilde{s}_1(t)s_2(t) - s_1(t)\tilde{s}_2(t)}{s_1(t)s_2(t) + \tilde{s}_1(t)\tilde{s}_2(t)} \ . \tag{8}$$

Let us consider two examples.

Example 1. Two coupled Rössler oscillators. The equations of the coupled system are

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + 0.15y_{1,2}, \\ \dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10), \end{aligned} \tag{9}$$

where parameters $\omega_1 = 0.89$ and $\omega_2 = 0.85$ define the average frequency of oscillations. They have been chosen in order to work within the frequency region without large periodic windows. To generate the signals with slowly varying parameters, we modulate coupling coefficient $\varepsilon = 0.03 + 0.02\sin(0.01t)$, and calculate the relative phase between $x_1(t)$ and $x_2(t)$. The results are shown in Fig 3. Due to modulation of the coupling, oscillators synchronize and desynchronize repeatedly. From these bivariate data we can easily distinguish time intervals, where the phase difference is constant, i.e. phases are locked. Respectively, we can conclude that within these intervals there is a resonant interaction between the systems, and they are synchronized.

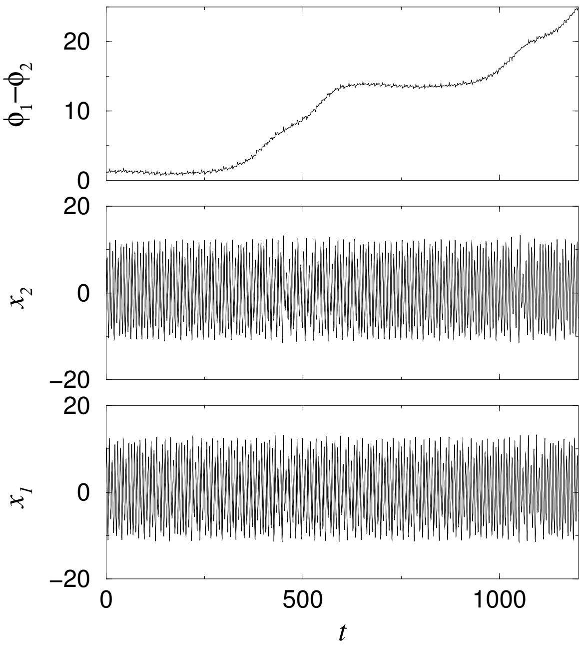

Fig. 3. Phase difference between two coupled Rössler oscillators. The coupling coefficient changes slowly with time. The periods of synchronous motion can be clearly seen

Example 2. Coupled Rössler and van der Pol oscillators. Similar results are found if two completely different systems, namely periodic van der Pol oscillator and chaotic Rössler systems, are coupled:

$$\begin{aligned}
\dot{x} &= -\omega_1 y - z + \varepsilon(u - x),\\
\dot{y} &= \omega_1 x + 0.15y,\\
\dot{z} &= 0.2 + z(x - 10),\\
\ddot{u} &- \mu(1 - u^2)\dot{u} + \omega_2 u = \varepsilon(x - u)\ ,
\end{aligned} \tag{10}$$

where $\omega_1 = 0.85$, $\omega_2 = 0.85$, and $\varepsilon = 0.02 + 0.02\sin(0.01t)$. The results are presented in Fig 4.

5 Conclusions

We have described a consistent method of calculation of the phase difference between two time series. We have shown that this method can be effectively

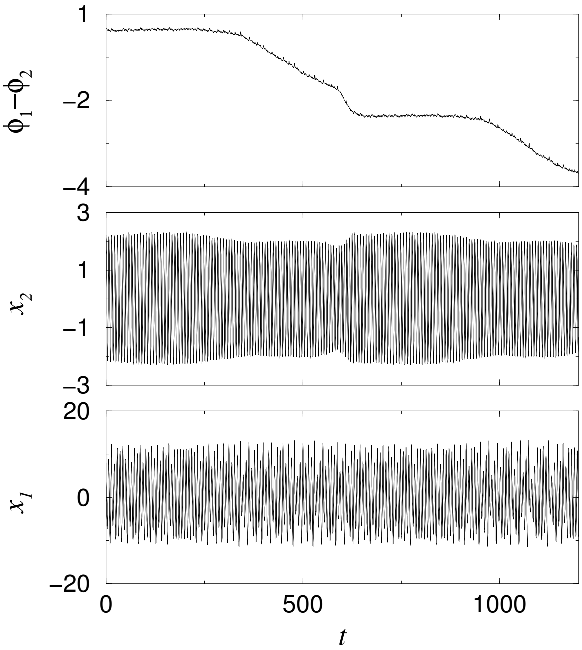

Fig. 4. Phase difference between coupled Rössler and van der Pol oscillators. The coupling coefficient changes slowly with time. The periods of synchronous motion can be clearly seen

used to reveal time-varying weak interaction between self-oscillating systems, which can be either chaotic or periodic.

Let us stress that if the phase difference between components of bivariate data is bounded, it does not necessary mean that the signals are generated by two synchronized oscillatory systems. For example, these signals can be the input and output of some phase-shifting (nonlinear) filter. Nevertheless, the technique can be formally applied; both the assumption on the underlying model and the interpretation of the result depends on the particular problem. This is similar to the usage of the coherence function and phase of the cross-spectra: although the model underlying cross-spectrum calculation is an one input – one output linear system, the technique can be applied to arbitrary bivariate data.

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